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Technical Report

On the Computation of Complex-valued Gradients with Application to Statistically Optimum Beamforming

by

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Abstract

This report describes the computation of gradients by algorithmic differentiation for statistically optimum beamforming operations. Especially the derivation of complex-valued functions is a key component of this approach. Therefore the real-valued algorithmic differentiation is extended via the complex-valued chain rule. In addition to the basic mathematic operations the derivative of the eigenvalue problem with complex-valued eigenvectors is one of the key results of this report. The potential of this approach is shown with experimental results on the CHiME-3 challenge database. There, the beamforming task is used as a front-end for an ASR system. With the developed derivatives a joint optimization of a speech enhancement and speech recognition system w.r.t. the recognition optimization criterion is possible.

Contents

1 Introduction

Over the years neural networks (NNs) have become state-of-the-art in many speech signal processing, e.g. automatic speech recognition (ASR). Recently they also gained impact in speech enhancement. There a NN estimates masks which are used to calculate the power spectral density (PSD) matrices of the speech and noise signal [Erd+16; Hey+15; HDHU16]. With these PSD matrices the beamforming coefficients are calculated with a statistical model based approach. Used with a strong ASR backend this system can improve the performance notably.

Looking at the complete system containing both speech enhancement and ASR, it is suboptimal to have two separate NNs involved. Both are trained with different objective functions which does not ensure, that the mask estimation is optimal in terms of speech recognition. To overcome this problem the gradients of the model-based beamforming operations have to be known to use the backpropagation algorithm for optimizing NNs. Since the beamformer operates with the short-time Fourier transform (STFT) of the noisy signal complex-valued operations are involved. This requires complex-valued gradients of many operations, including the eigendecomposition.

The objective function used in this report is the output signal-to-noise ratio (SNR) of the beamformer, which is closer related to speech enhancement than the mask purity criterion. The gradients are calculated using algorithmic differentiation (AD), where a function is decomposed into a chain of elementary functions of which the gradient w.r.t. the desired quantity can be calculated. In especially we extend many well-known real-valued gradients to their complex-valued case. Therefore different concepts of deriving those gradients are presented. Then some examples are given, including the gradient of the discrete fourier transformation (DFT), the matrix inverse and the complex-valued eigenvalue problem. Also the gradient of the Cholesky decomposition is given.

In Chapter 2 the system consisting of a beamformer and ASR part is shown. Also the statistically beamforming operations are described. The complex-valued backpropagation, gradient descent and the derivation of the backward gradient are explained in Chapter 3. Chapter 4 shows the derivation of the backward gradients for many commonly used complex-valued operations. An overview of all gradients can be found in Section 4.18. The evaluation of the system with the derived gradients is done in Chapter 5. Chapter 6 summarizes the results and gives a short outlook.

2 Model

2.1 Description of the system

The system contains a D-dimensional microphone array, that receives a noisy speech signal

$$\mathbf{Y}_{f,t} = \mathbf{X}_{f,t} + \mathbf{N}_{f,t}, \tag{2.1}$$

where the D channels are combined to an observation vector. $\mathbf{Y}_{f,t}$, $\mathbf{X}_{f,t}$ and $\mathbf{N}_{f,t}$ are the STFT of noisy speech, clean speech and noise at frequency bin f and time frame t, respectively. The system is designed to extract the clean speech signal $X_{f,t,1}$ observed by the first microphone. Here acoustic beamforming with the beamformer coefficient vector $\mathbf{w}_f$, which is assumed to be time-invariant, is used [Li+15, Ch. 10]:

$$\hat{X}_{f,t,1} = \mathbf{w}_f^{\mathrm{H}} \cdot \mathbf{Y}_{f,t}. \tag{2.2}$$

To obtain the beamformer coefficient vector the speech and noise PSD matrices have to be estimated. This is done via a NN, which is applied to each microphone channel separately. It estimates a speech mask $M_{f,t,d}^{(X)}$ and a noise mask $M_{f,t,d}^{(N)}$ for each channel. After combining (i.e. averaging) the D speech and noise masks to a single speech and a single noise mask and assuming a zero-mean observation signal $\mathbf{Y}_{f,t}$ the PSD matrices can be estimated with a maximum likelihood (ML) estimator [Bis06, Eq. 9.19]

$$\mathbf{\Phi}_{\nu\nu f} = \frac{\sum_{t=1}^{T} \left(\sum_{d=1}^{D} M_{f,t,d}^{(\nu)} \right) \mathbf{Y}_{f,t} \cdot \mathbf{Y}_{f,t}^{\mathrm{H}}}{\sum_{t=1}^{T} \left(\sum_{d=1}^{D} M_{f,t,d}^{(\nu)} \right)}, \tag{2.3}$$

where $\nu \in \{X, N\}$ and where $(\cdot)^{\mathrm{H}}$ denotes the conjugate transpose. With these estimated PSD matrices different statistically optimal beamformers can be computed (see Section 2.2).

An objective function calculated from the beamformer output is used to update the parameters of the NN. In this report the negative SNR of the normalized beamformer output signal

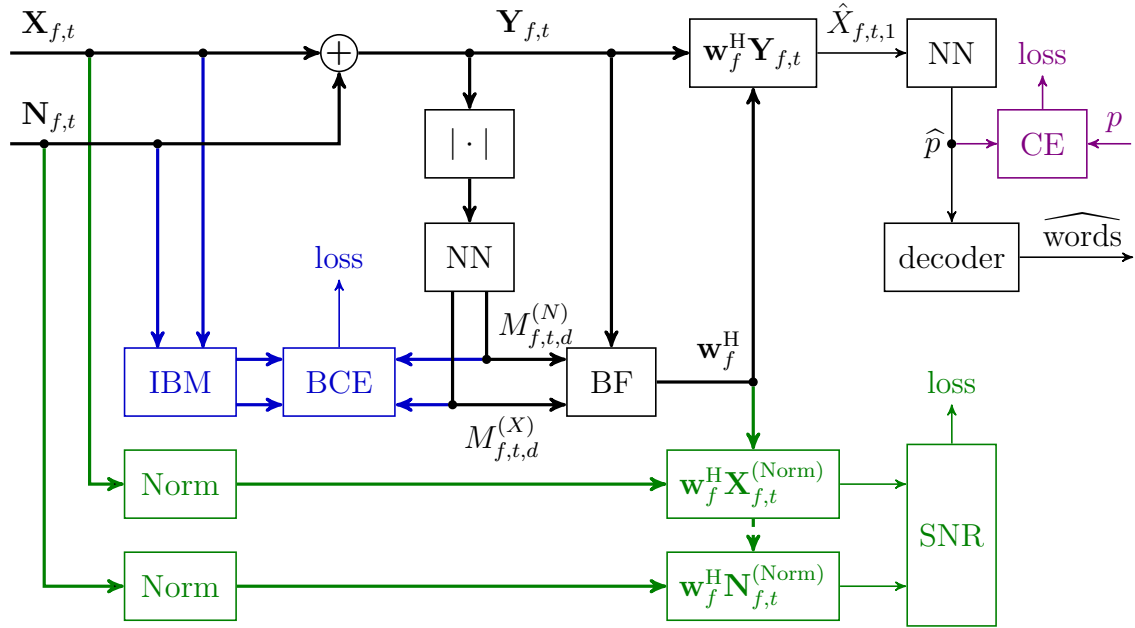$$J = -10 \cdot \log_{10} \frac{P^{(X)}}{P^{(N)}}, \tag{2.4}$$

Figure 2.1: Block diagram of the training system of the described system with different training objectives in colors

with the power of the beamformed signal

$$P^{(\nu)} = \frac{1}{T} \sum_{t=1}^{T} \sum_{f=1}^{F} |\mathbf{w}_f^{\mathrm{H}} \cdot \boldsymbol{\nu}_{f,t}^{(\mathrm{Norm})}|^2, \tag{2.5}$$

applied to the normalized clean speech and noise signal

$$\boldsymbol{\nu}_{f,t}^{(\mathrm{Norm})} = \frac{\boldsymbol{\nu}_{f,t}}{\sqrt{\sum_{\check{t}=1}^{T} ||\boldsymbol{\nu}_{f,\check{t}}||^2}}, \tag{2.6}$$

is used as the objective function. T and F are the total number of frames and frequency bins, respectively. Every frequency contributes equally strong to the objective due to the normalization.

Fig. 2.1 gives an overview of the considered NN supported acoustic beamformer for speech enhancement and the following ASR with all three used objective functions [Hey+15]. The different colors mark the different objective functions which are used for training. After both networks are trained only the black parts are needed to decode the spoken word sequence. In [Hey+15; HDHU16] the mask estimation NN uses heuristic ideal binary masks (IBMs) as targets and binary cross entropy (BCE) as the objective function. This is shown in blue. Our new approach (shown in green) is to use the negative SNR of the beamformed signal as the objective function and propagate the gradient through the beamformer back to the mask estimation NN. The global approach is to use a common objective function for both front-end and back-end networks. For the latter the cross entropy is used as the objective function which is well known for ASR tasks (here shown in purple) and the gradient is propagated back to the mask

estimation NN for optimization w.r.t. the cross entropy. The reference target p is the framewise label/state posterior probability or sequence probability (more information in [Hey+17]).

The computation of the objective function J involves complex-valued quantities although the objective is real-valued. Thus complex-valued gradients are needed for error backpropagation, which are described in Chapter 3.

2.2 Statistically optimal beamforming operations

In this section different beamforming operations are presented. First the widely used minimum variance distortionless response (MVDR) beamformer is described, followed by the generalized eigenvector (GEV) beamformer, which maximizes the SNR of the output signal.

2.2.1 MVDR beamformer

The MVDR beamformer tries to minimizes the variance of the noise power without distorting the signal from the desired direction. The coefficients can be calculated via

$$\mathbf{w}_f^{(\text{MVDR})} = \frac{\boldsymbol{\Phi}_{NNf}^{-1} \cdot \mathbf{w}_f^{(\text{PCA})}}{\mathbf{w}_f^{(\text{PCA})\text{H}} \cdot \boldsymbol{\Phi}_{NNf}^{-1} \cdot \mathbf{w}_f^{(\text{PCA})}}, \tag{2.7}$$

where $\mathbf{w}_f^{(\text{PCA})} = \mathcal{P}(\boldsymbol{\Phi}_{XXf})$ is the eigenvector corresponding to the largest eigenvalue of $\boldsymbol{\Phi}_{XXf}$ [Hay02]. The gradients needed for forward and backward mode AD of the MVDR beamformer can be obtained with the examples found in chapter 4.

2.2.2 GEV beamformer

The GEV beamforming vector [WHU06]

$$\mathbf{w}_f^{(\text{GEV})} = \underset{\mathbf{w}_f}{\operatorname{argmax}} \left\{ \frac{\mathbb{E}\{|\mathbf{w}_f^{\text{H}}\mathbf{Y}_{f,t}|^2 \mid \mathbf{N}_{f,t} = 0\}}{\mathbb{E}\{|\mathbf{w}_f^{\text{H}}\mathbf{Y}_{f,t}|^2 \mid \mathbf{X}_{f,t} = 0\}} \right\} = \underset{\mathbf{w}_f}{\operatorname{argmax}} \left\{ \frac{\mathbf{w}_f^{\text{H}}\boldsymbol{\Phi}_{XXf}\mathbf{w}_f}{\mathbf{w}_f^{\text{H}}\boldsymbol{\Phi}_{NNf}\mathbf{w}_f} \right\}, \tag{2.8}$$

also known as max SNR beamfomrer, requires the solution of the generalized eigenvalue problem

$$\boldsymbol{\Phi}_{XXf}\mathbf{W}_f = \boldsymbol{\Phi}_{NNf}\mathbf{W}_f\boldsymbol{\Lambda}_f, \tag{2.9}$$

where $\mathbf{w}_f^{(\text{GEV})}$ is the eigenvector corresponding to the largest eigenvalue. Under the assumption that the inverse of $\boldsymbol{\Phi}_{NNf}$ exists, the generalized eigenvalue problem for

hermitian matrices is equal to a normal eigenvalue problem

$$\mathbf{\Phi}_{NN_f}^{-1}\mathbf{\Phi}_{XX_f}\mathbf{W}_f = \mathbf{W}_f\mathbf{\Lambda}_f, \qquad (2.10)$$

$$\mathbf{\Phi}_f\mathbf{W}_f = \mathbf{W}_f\mathbf{\Lambda}_f, \qquad (2.11)$$

where usually $\mathbf{\Phi}_f$ is a non hermitian matrix. This formulation is required to obtain the gradient, which is explained in the following chapters.

3 Complex-valued backpropagation

In the forward step of a NN an objective from the input data and the network parameters, which should be optimized, is calculated. Then the backpropagation algorithm uses the chain rule to calculate the gradients of the objective function w.r.t. all adjustable parameters. With these gradients an optimizer (e.g. gradient decent) updates the parameters. The real-valued gradient calculation rules, especially the chain rule

$$\frac{\partial J}{\partial x} = \frac{\partial J}{\partial \omega}\frac{\partial \omega}{\partial x}, \tag{3.1}$$

are well known. This gets more difficult if complex numbers are involved in the forward step. There is a definition for complex differentiation: A function is complex differentiable (holomorph) if the real and imaginary part satisfy the Cauchy–Riemann equations [Kre10, Chapter 13]. Since many relevant complex function are not holomorph (e.g. absolute value, real/imaginary part and conjugation) we dismiss this approach.

An alternative definition for the complex derivative is described in [Bra83] the *Wirtinger calculus*

$$\frac{\partial J}{\partial z^*} = \frac{1}{2}\left(\frac{\partial J}{\partial x} + \mathrm{j}\frac{\partial J}{\partial y}\right), \tag{3.2}$$

$$\frac{\partial J}{\partial z} = \frac{1}{2}\left(\frac{\partial J}{\partial x} - \mathrm{j}\frac{\partial J}{\partial y}\right), \tag{3.3}$$

where J is the real-valued objective, x and y are real numbers and $\frac{\partial J}{\partial x}$ is the partial derivative of J w.r.t. x. This definition is well known for complex optimization problems. It can be interpreted as a pure real-valued problem, because the objective function is real-valued and all complex operations can be separated in real operations. Nevertheless in this report we use the *Wirtinger calculus* to derive rules for calculating complex gradients instead of defining the forward step via real-valued operations.

Since an objective function has to be real valued it is easy to switch between (3.2) and (3.3)

$$\frac{\partial J}{\partial z^*} = \left(\frac{\partial J}{\partial z}\right)^*. \tag{3.4}$$

Although both definitions can be used we mainly focus on (3.2) here.

3.1 Chain rule

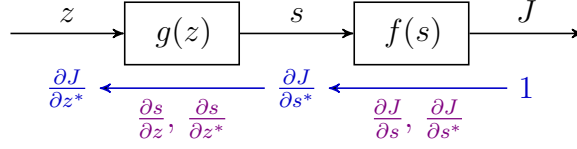Figure 3.1: Visualization of the chain rule: forward step, backward gradient and intermediate derivatives

The most important rule in this work is the chain rule. This rule describes the relation between gradients, when a function is applied to the output of another. Therefore, we start with a chain of two functions (see Fig. 3.1)

$$
\begin{aligned}
J &= f(s) \quad \text{where } s = \sigma + \mathrm{j}\omega, \\
s &= g(z) \quad \text{where } z = x + \mathrm{j}y, \\
s, z &\in \mathbb{C}, \quad J, \sigma, \omega, x, y \in \mathbb{R}.
\end{aligned}
\tag{3.5}
$$

Combining the real-valued chain rule [Kre10, p. 393]

$$
\frac{\partial J}{\partial x} = \frac{\partial J}{\partial \sigma}\frac{\partial \sigma}{\partial x} + \frac{\partial J}{\partial \omega}\frac{\partial \omega}{\partial x}
\tag{3.6}
$$

and (3.2) leads to

$$
\begin{aligned}
\frac{\partial J}{\partial z^*} &= \frac{1}{2}\left(\frac{\partial J}{\partial \sigma}\frac{\partial \sigma}{\partial x} + \frac{\partial J}{\partial \omega}\frac{\partial \omega}{\partial x} + \mathrm{j}\frac{\partial J}{\partial \sigma}\frac{\partial \sigma}{\partial y} + \mathrm{j}\frac{\partial J}{\partial \omega}\frac{\partial \omega}{\partial y}\right) \\
&= \dots^{1} = \left(\frac{\partial J}{\partial s^*}\right)^{*}\frac{\partial s}{\partial z^*} + \frac{\partial J}{\partial s^*}\left(\frac{\partial s}{\partial z}\right)^{*},
\end{aligned}
\tag{3.7}
$$

where

$$
\begin{aligned}
\frac{\partial s}{\partial z^*} &= \frac{1}{2}\left(\frac{\partial s}{\partial x} + \mathrm{j}\frac{\partial s}{\partial y}\right) = \frac{1}{2}\left(\frac{\partial \sigma}{\partial x} + j\frac{\partial \omega}{\partial x} + \mathrm{j}\frac{\partial \sigma}{\partial y} - \frac{\partial \omega}{\partial y}\right), \\
\frac{\partial s}{\partial z} &= \frac{1}{2}\left(\frac{\partial s}{\partial x} - \mathrm{j}\frac{\partial s}{\partial y}\right) = \frac{1}{2}\left(\frac{\partial \sigma}{\partial x} + j\frac{\partial \omega}{\partial x} - \mathrm{j}\frac{\partial \sigma}{\partial y} + \frac{\partial \omega}{\partial y}\right).
\end{aligned}
\tag{3.8}
$$

This solution for the chain rule is the same as in [KD09; Ami+11; BT11; Wik16]

3.2 Reuse of a variable

Sometimes a variable is used more than one time for the calculation of the objective function. Thus this variable receives more than one gradient which are accumulated like in the real-valued case.

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[1] A detailed calculation can be found in Appendix A.2

With this rule we can extend the scalar chain rule to tensors with arbitrary dimensions. When $\mathbf{S}$ and $\mathbf{Z}$ are tensors

$$\mathbf{S} \in \mathbb{C}^{A_1 \times \cdots \times A_N} \ , \quad \mathbf{Z} \in \mathbb{C}^{B_1 \times \cdots \times B_M}$$

the chain rule can be denoted elementwise

$$\frac{\partial J}{\partial Z^*_{b_1,\dots,b_M}} = \sum_{a_1,\dots,a_N} \left(\frac{\partial J}{\partial S^*_{a_1,\dots,a_N}}\right)^* \frac{\partial S_{a_1,\dots,a_N}}{\partial Z^*_{b_1,\dots,b_M}} + \frac{\partial J}{\partial S^*_{a_1,\dots,a_N}} \left(\frac{\partial S_{a_1,\dots,a_N}}{\partial Z_{b_1,\dots,b_M}}\right)^*, \qquad (3.9)$$

where a_n and b_m ($n \in \{1,\dots,N\}$ and $m \in \{1,\dots,M\}$) indicate the elements in the n-th and m-th dimension, respectively.

A special case of this chain rule, where only $\mathbf{S}$ is a two dimensional matrix and z is scalar, can be written as

$$\frac{\partial J}{\partial z^*} = \sum_{n,m} \left(\left(\frac{\partial J}{\partial S^*_{n,m}}\right)^* \frac{\partial S_{n,m}}{\partial z^*} + \frac{\partial J}{\partial S^*_{n,m}} \left(\frac{\partial S_{n,m}}{\partial z}\right)^*\right)$$

$$= \mathrm{Tr}\left(\left(\frac{\partial J}{\partial \mathbf{S}^*}\right)^{\mathrm{H}} \frac{\partial \mathbf{S}}{\partial z^*} + \left(\frac{\partial J}{\partial \mathbf{S}^*}\right)^{\mathrm{T}} \left(\frac{\partial \mathbf{S}}{\partial z}\right)^*\right), \qquad (3.10)$$

where $\mathrm{Tr}\left(\cdot\right)$ is the sum of the diagonal elements, $\frac{\partial J}{\partial \mathbf{S}^*}$ the derivative of J w.r.t. each element of $\mathbf{S}$ and $\frac{\partial \mathbf{S}}{\partial z^*}$ the derivative of each element of $\mathbf{S}$. Both gradient matrices have the same shape as $\mathbf{S}$. This notation using the $\mathrm{Tr}\left(\cdot\right)$ operation is only useful for symbolic calculations in an algorithm explained later.

To compare this chain rule with [DRHU16, Eq. 7], $\mathbf{s}$ and $\mathbf{z}$ have to be vectors

$$\frac{\partial J}{\partial \mathbf{z}^*} = \left(\left(\frac{\partial J}{\partial \mathbf{s}^*}\right)^{\mathrm{H}} \frac{\partial \mathbf{s}}{\partial \mathbf{z}^*} + \left(\frac{\partial J}{\partial \mathbf{s}^*}\right)^{\mathrm{T}} \left(\frac{\partial \mathbf{s}}{\partial \mathbf{z}}\right)^*\right)^{\mathrm{T}}, \qquad (3.11)$$

$$\frac{\partial \mathbf{s}}{\partial \mathbf{z}^*} = \begin{bmatrix} \frac{\partial s_1}{\partial z_1^*} & \frac{\partial s_1}{\partial z_2^*} & \cdots \\ \frac{\partial g_2}{\partial z_1^*} & \frac{\partial s_2}{\partial z_2^*} & \\ \vdots & & \ddots \end{bmatrix}. \qquad (3.12)$$

3.3 Real-valued input

One important case of a complex operation is a real-valued input. This means, that the imaginary part $\Im\{z\} = y$ is zero, so s is independent of it. Therefore the gradient w.r.t. y has to be zero and (3.7) simplifies to

$$\frac{\partial J}{\partial z^*} = \frac{1}{2}\left(\frac{\partial J}{\partial \sigma}\frac{\partial \sigma}{\partial x} + \frac{\partial J}{\partial \omega}\frac{\partial \omega}{\partial x} + \mathrm{j}\frac{\partial J}{\partial \sigma}0 + \mathrm{j}\frac{\partial J}{\partial \omega}0\right)$$

$$= \frac{1}{2}\left(\frac{\partial J}{\partial \sigma}\frac{\partial \sigma}{\partial x} + \frac{\partial J}{\partial \omega}\frac{\partial \omega}{\partial x}\right)$$

$$= \Re\left\{\left(\frac{\partial J}{\partial s^*}\right)^* \frac{\partial s}{\partial z^*} + \frac{\partial J}{\partial s^*}\left(\frac{\partial s}{\partial z}\right)^*\right\}, \qquad (3.13)$$

where $\Re\{\cdot\}$ denotes the real part. Note that the real part operator has to be applied to the gradient if the input is real-valued, because in general the gradient is complex-valued.

3.4 Gradient descent

Optimizing the parameters of a NN can be done with gradient descent. For the real-valued case this is done via

$$x^{(\text{new})} = x^{(\text{old})} - \frac{\mu}{2}\frac{\partial J}{\partial x}\bigg|_{x=x^{(\text{old})}}, \tag{3.14}$$

$$y^{(\text{new})} = y^{(\text{old})} - \frac{\mu}{2}\frac{\partial J}{\partial y}\bigg|_{y=y^{(\text{old})}}, \tag{3.15}$$

where $\mu/2$ is the stepsize. Combining both equations with the definition of complex numbers and the *Wirtinger calculus* (3.2)

$$\begin{aligned}
z^{(\text{new})} &= x^{(\text{new})} + \mathrm{j}y^{(\text{new})} \\
&= x^{(\text{old})} - \frac{\mu}{2}\frac{\partial J}{\partial x}\bigg|_{x=x^{(\text{old})}} + \mathrm{j}y^{(\text{old})} - \frac{\mu}{2}\frac{\partial J}{\partial y}\bigg|_{y=y^{(\text{old})}} \\
&= x^{(\text{old})} + \mathrm{j}y^{(\text{old})} - \frac{\mu}{2}\left(\frac{\partial J}{\partial x}\bigg|_{x=x^{(\text{old})}} + \mathrm{j}\frac{\partial J}{\partial y}\bigg|_{y=y^{(\text{old})}}\right) \\
&= z^{(\text{old})} - \mu\frac{\partial J}{\partial z^*}\bigg|_{z=z^{(\text{old})}} \tag{3.16} \\
&= z^{(\text{old})} - \mu\left(\frac{\partial J}{\partial z}\right)^*\bigg|_{z=z^{(\text{old})}} \tag{3.17}
\end{aligned}$$

results in a solution for the complex gradient descent. Note, that there are two possibilities to calculate the gradient. The first one, shown in (3.17), is to determine $\frac{\partial J}{\partial z}$ and use its conjugate complex as the update like it is done in the framework [Mac+15]. Otherwise $\frac{\partial J}{\partial z^*}$ is determined and the same update rule as in the real-valued case is used, denoted in (3.16).

3.5 Derivation of the backward gradient

The gradient of a function is needed for the training of a NN with the backpropagation method. Here this is done via AD where a function is segmented into a chain of elementary operations. The input gradient of each elementary operation is calculated with the chain rule from the output gradient. This is done to get the gradient w.r.t. the target quantities. This method is called building block concept.

The idea is to build a computational graph for the complete network. Beginning at the end, each block receives the output gradient and calculates the input gradient using the

chain rule until the gradient of each weight is known. The building block system can also be used to analytically combine intermediate blocks to one block with a numerically more stable gradient and an improved execution time.

In this section different strategies to derive the gradient of a function are presented. First, two analytical derivations are described followed by a numerical approach. The numerical approach is used to verify the implementation of the analytical gradients. These strategies are only two suggestions to get a solution. There may be others which are better suited for different tasks.

3.5.1 Conventional derivation

Following the ideas from [DRHU16] to derive the gradient of a complex valued function is intuitive. After reformulating the function to a scalar version it is possible to calculate the derivative of each output w.r.t. each input $(\frac{\partial s_n}{\partial z_m}, \frac{\partial s_n}{\partial z_m^*})$. Sometimes it is helpful to use the real-valued derivatives $(\frac{\partial \sigma}{\partial x}, \frac{\partial \sigma}{\partial y}, \frac{\partial \omega}{\partial x}, \frac{\partial \omega}{\partial y})$. Now the derivatives can be obtained by inserting the derivatives in (3.9) or (3.11) in the case of vectors.

This approach, summarized in Alg. 1, to calculate the derivative is useful when the function of interest describes scalar operation. However a disadvantage is that a non implicit function is required, i.e. the target has to be alone on one side of the equal sign. The algorithm in the following section eliminates this drawback.

Algorithm 1 Conventional derivation

1: Calculate the derivative of each output w.r.t. each input: $\frac{\partial s_n}{\partial z_m}, \frac{\partial s_n}{\partial z_m^*}$
2: Insert in (3.9) or (3.11) in the case of vectors

3.5.2 Derivation over forward derivative

This complex-valued approach as a generalization of [Gil08] differs from the previous section. Here the initial equation is a matrix operation $\mathbf{C} = \mathbf{h}(\mathbf{A}, \mathbf{B})$, where $\mathbf{C}$ is the output and $\mathbf{A}$ and $\mathbf{B}$ are the inputs. The equation is reformulated to a simple characteristic equation which describes the dependencies between the matrices, e.g. $\mathbf{AC} = \mathbf{B}$ is a simpler formulation as $\mathbf{C} = \mathbf{A}^{-1}\mathbf{B}$ (Section 4.15). Now both sides of the equation are derived w.r.t. $\zeta \in \{z, z^*\}$ where ζ is assumed as the input variable before this operation. With the reformulation of this equation to $\frac{\partial \mathbf{C}}{\partial \zeta} = \dots$ the forward mode AD is obtained [Gil08] which is not required for the backpropagation algorithm.[2]

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[2] The forward mode AD calculates the gradient by starting at the input and propagate through all calculations. Therefore it requires for each parameter an independent calculation of the gradient. In many applications this is too expensive and the forward mode AD ($\frac{\partial \mathbf{C}}{\partial z^*}$) is used as an analytical helper to receive the backward mode AD ($\frac{\partial J}{\partial \mathbf{A}^*}$). Note: In forward mode the derivation of the output $\mathbf{C}$ w.r.t. the input z^* and in backward mode the derivation of the objective function w.r.t. the input $\mathbf{A}$ is required.

To obtain the backward mode AD, needed for optimization, the forward mode AD is inserted in (3.10)

$$\frac{\partial J}{\partial z^*} = \text{Tr}\left(\left(\frac{\partial J}{\partial \mathbf{C}^*}\right)^{\text{H}} \frac{\partial \mathbf{C}}{\partial z^*} + \left(\frac{\partial J}{\partial \mathbf{C}^*}\right)^{\text{T}} \left(\frac{\partial \mathbf{C}}{\partial z}\right)^*\right) \tag{3.18}$$

and rearranged to

$$\frac{\partial J}{\partial z^*} = \text{Tr}\left((\ldots)^{\text{H}} \frac{\partial \mathbf{A}}{\partial z^*} + (\ldots)^{\text{T}} \left(\frac{\partial \mathbf{A}}{\partial z}\right)^* + (\ldots)^{\text{H}} \frac{\partial \mathbf{B}}{\partial z^*} + (\ldots)^{\text{T}} \left(\frac{\partial \mathbf{B}}{\partial z}\right)^*\right). \tag{3.19}$$

A comparison of the coefficients with

$$\frac{\partial J}{\partial z^*} = \text{Tr}\left(\left(\frac{\partial J}{\partial \mathbf{A}^*}\right)^{\text{H}} \frac{\partial \mathbf{A}}{\partial z^*} + \left(\frac{\partial J}{\partial \mathbf{A}^*}\right)^{\text{T}} \left(\frac{\partial \mathbf{A}}{\partial z}\right)^* + \left(\frac{\partial J}{\partial \mathbf{B}^*}\right)^{\text{H}} \frac{\partial \mathbf{B}}{\partial z^*} + \left(\frac{\partial J}{\partial \mathbf{B}^*}\right)^{\text{T}} \left(\frac{\partial \mathbf{B}}{\partial z}\right)^*\right) \tag{3.20}$$

results in the backward mode AD $\frac{\partial J}{\partial \mathbf{A}^*}$ and $\frac{\partial J}{\partial \mathbf{B}^*}$. Therefore the use of the trace identities like

$$\text{Tr}\left(A^{\text{T}}\right) = \text{Tr}\left(A\right), \tag{3.21}$$
$$\text{Tr}\left(AB\right) = \text{Tr}\left(BA\right), \tag{3.22}$$
$$\text{Tr}\left(A + B\right) = \text{Tr}\left(A\right) + \text{Tr}\left(B\right) \tag{3.23}$$

is beneficial.

With this concept, described in Alg. 2, it is possible to find more solutions than with the previous one. For example the backward gradient of the inverse $\mathbf{C} = \mathbf{A}^{-1}$ (Section 4.14) can be obtained which can not be decomposed into a trivial element-wise operation.

Algorithm 2 Derivation over forward derivative

1: $\mathbf{C} = \mathbf{h}(\mathbf{A}, \mathbf{B})$ with matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$
2: Find a simple representation of the equation (Implicit functions are allowed)
3: Compute the derivative of both sides w.r.t. z^* and z
4: Reformulate and insert $\frac{\partial \mathbf{C}}{\partial z^*}$ and $\frac{\partial \mathbf{C}}{\partial z}$ in (3.10)
5: Use trace identities to reformulate
6: Compare the coefficients with:
   $\frac{\partial J}{\partial z*} = \text{Tr}\left(\left(\frac{\partial J}{\partial \mathbf{A}^*}\right)^{\text{H}} \frac{\partial \mathbf{A}}{\partial z^*} + \left(\frac{\partial J}{\partial \mathbf{A}^*}\right)^{\text{T}} \left(\frac{\partial \mathbf{A}}{\partial z}\right)^*\right) + \text{Tr}\left(\left(\frac{\partial J}{\partial \mathbf{B}^*}\right)^{\text{H}} \frac{\partial \mathbf{B}}{\partial z^*} + \left(\frac{\partial J}{\partial \mathbf{B}^*}\right)^{\text{T}} \left(\frac{\partial \mathbf{B}}{\partial z}\right)^*\right)$
7: The comparison of $\left(\frac{\partial J}{\partial \mathbf{A}^*}\right)^{\text{H}}$ and $\left(\frac{\partial J}{\partial \mathbf{A}^*}\right)^{\text{T}}$ should result in the same solution for $\frac{\partial J}{\partial \mathbf{A}^*}$

3.5.3 Numeric derivative [Tok+15]

This concept can be used to verify the analytical solutions. The definition of the derivative is used to calculate an approximation for the derivative. Therefore only the

forward step needs to be known so that the algorithm can approximate the derivative for a specific input value. The Algorithm 3 is very slow but a good way to verify the correctness of the analytical solution. The operator vec in the Algorithm 3 is the vectorize or flattern operator. It takes the input with arbitrary numbers of dimensions and concatenate all elements to a vector. The new order of the elements is not necessary in this example.

Algorithm 3 Numerical derivative

1: $\mathbf{s} = \mathbf{g}(z) \leftarrow \text{vec}\{\mathbf{g}(z)\}$

2: $\frac{\partial \mathbf{s}}{\partial x} \approx \frac{\mathbf{g}(x+\varepsilon+\mathrm{j}y)-\mathbf{g}(x-\varepsilon+\mathrm{j}y)}{2\varepsilon}$

3: $\frac{\partial \mathbf{s}}{\partial y} \approx \frac{\mathbf{g}(x+\mathrm{j}(y+\varepsilon))-\mathbf{g}(x+\mathrm{j}(y-\varepsilon))}{2\varepsilon}$

4: $\frac{\partial J}{\partial z^*} \approx \left(\frac{\partial J}{\partial \mathbf{s}^*}\right)^{\mathrm{H}} \frac{1}{2}\left(\frac{\partial \mathbf{s}}{\partial x} + \mathrm{j}\frac{\partial \mathbf{s}}{\partial y}\right) + \left(\frac{\partial J}{\partial \mathbf{s}^*}\right)^{\mathrm{T}} \left(\frac{1}{2}\left(\frac{\partial \mathbf{s}}{\partial x} - \mathrm{j}\frac{\partial \mathbf{s}}{\partial y}\right)\right)^*$

---

4 Blocks

In this chapter we present some examples for the complex-valued backpropagation. Starting with some fundamental elementwise operations it finally ends with the derivation of the eigenvalue problem. All described operations are tested against a numerical gradient. Many functions were developed for [DRHU16], but they are not described there. The derivation of the matrix operations are inspired by their real-valued counterparts in [Gil08]. Also the solutions are compared with their real-valued implementation in [Tok+15] and complex-valued implementation in [Mac+15] in case they exists.

4.1 Identity

The simplest possible example is the identity

$$s = g(z) = z. \tag{4.1}$$

With the derivatives

$$\frac{\partial s}{\partial z} = 1 \qquad \text{and} \qquad \frac{\partial s}{\partial z^*} = 0$$

and (3.7) the gradient is

$$\frac{\partial J}{\partial z^*} = \left(\frac{\partial J}{\partial s^*}\right)^* 0 + \frac{\partial J}{\partial s^*}(1)^* = \frac{\partial J}{\partial s^*}. \tag{4.2}$$

As expected the identity does not change the gradient.

4.2 Conjugation

An important operation for complex numbers is the conjugation

$$s = g(z) = z^*. \tag{4.3}$$

The derivatives are again as trivial as for the identity

$$\frac{\partial s}{\partial z} = 0 \qquad \text{and} \qquad \frac{\partial s}{\partial z^*} = 1$$

and with (3.7) the gradient is

$$\frac{\partial J}{\partial z^*} = \left(\frac{\partial J}{\partial s^*}\right)^* 1 + \frac{\partial J}{\partial s^*}(0)^* = \left(\frac{\partial J}{\partial s^*}\right)^*. \tag{4.4}$$

In case that z is real-valued the gradient $\frac{\partial J}{\partial s^*}$ is real-valued, too, and the conjugation merges into the identity as expected.

4.3 Addition

Let

$$s = g(z_1, z_2) = z_1 + z_2, \tag{4.5}$$

then the gradients are

$$\frac{\partial J}{\partial z_1^*} = \frac{\partial J}{\partial s^*} \qquad \text{and} \qquad \frac{\partial J}{\partial z_2^*} = \frac{\partial J}{\partial s^*} \tag{4.6}$$

like the real-valued gradient.

4.4 Multiplication

In the case that

$$s = g(z_1, z_2) = z_1 \cdot z_2 \tag{4.7}$$

the derivatives are

$$\frac{\partial s}{\partial z_1} = z_2 \qquad \text{and} \qquad \frac{\partial s}{\partial z_1^*} = 0,$$

$$\frac{\partial s}{\partial z_2} = z_1 \qquad \text{and} \qquad \frac{\partial s}{\partial z_2^*} = 0$$

and with (3.7) the gradients are

$$\frac{\partial J}{\partial z_1^*} = \left(\frac{\partial J}{\partial s^*}\right)^* 0 + \frac{\partial J}{\partial s^*} z_2^* = \frac{\partial J}{\partial s^*} z_2^*, \tag{4.8}$$

$$\frac{\partial J}{\partial z_2^*} = \left(\frac{\partial J}{\partial s^*}\right)^* 0 + \frac{\partial J}{\partial s^*} z_1^* = \frac{\partial J}{\partial s^*} z_1^*. \tag{4.9}$$

4.5 Exponentiation

The square

$$s = g(z) = z^2 \tag{4.10}$$

is a special case of the Multiplication ($z = z_1 = z_2$). Therefore the gradient is

$$\frac{\partial J}{\partial z^*} = \frac{\partial J}{\partial s^*} z_1^* + \frac{\partial J}{\partial s^*} z_2^* = 2\frac{\partial J}{\partial s^*} z^*.$$

For the generalization $s = g(z) = z^n$ we split the equation in $s = z_1 z_2$ n-times, where $z_1 = z$ and $z_2 = z^{n-1}$. With the gradient of the multiplication

$$\frac{\partial J}{\partial z_1^*} = \frac{\partial J}{\partial s^*} z_2^* = \frac{\partial J}{\partial s^*} \left(z^*\right)^{n-1}$$

and the knowledge of Section 3.2 to accumulate all n gradients, we get

$$\frac{\partial J}{\partial z^*} = \frac{\partial J}{\partial s^*} n \left(z^*\right)^{n-1}. \tag{4.11}$$

4.6 Division

The derivation of the division is more complicated. Therefore define

$$s = g(z_1, z_2) = z_1/z_2 = z_1 z_3 \tag{4.12}$$

where $z_3 = 1/z_2$. We know from multiplication

$$\frac{\partial J}{\partial z_1^*} = \frac{\partial J}{\partial s^*} z_3^* = \frac{\partial J}{\partial s^*} \frac{1}{z_2^*} \tag{4.13}$$

and

$$\frac{\partial J}{\partial z_3^*} = \frac{\partial J}{\partial s^*} z_1^*.$$

From exponentiation with $n = -1$ we know that

$$\frac{\partial J}{\partial z_2^*} = \frac{\partial J}{\partial z_3^*} \left(-1\right) \left(z_2^*\right)^{-2}$$

Therefore the gradient is

$$\frac{\partial J}{\partial z_2^*} = \frac{\partial J}{\partial s^*} \frac{-z_1^*}{\left(z_2^*\right)^2}. \tag{4.14}$$

4.7 Absolute value

To derive the gradient of the absolute value

$$s = g(z) = |z| \in \mathbb{R} \tag{4.15}$$

we use the real derivatives with $z = x + \mathrm{j}y = |z|\, \mathrm{e}^{\mathrm{j}\varphi}$

$$\frac{\partial s}{\partial x} = \frac{\partial \sqrt{x^2 + y^2}}{\partial x} = \frac{x}{s} \qquad \text{and} \qquad \frac{\partial s}{\partial y} = \frac{\partial \sqrt{x^2 + y^2}}{\partial y} = \frac{y}{s}$$

and insert them in the definition (3.8)

$$\frac{\partial s}{\partial z^*} = \frac{1}{2}\left(\frac{\partial s}{\partial x} + \mathrm{j}\frac{\partial s}{\partial y}\right) = \frac{z}{s} = \frac{1}{2}\,\mathrm{e}^{\mathrm{j}\varphi},$$

$$\frac{\partial s}{\partial z} = \frac{1}{2}\left(\frac{\partial s}{\partial x} - \mathrm{j}\frac{\partial s}{\partial y}\right) = \frac{z}{s} = \frac{1}{2}\,\mathrm{e}^{-\mathrm{j}\varphi}\,.$$

Using (3.7) leads to

$$\frac{\partial J}{\partial z^*} = \underbrace{\left(\frac{\partial J}{\partial s^*}\right)^*}_{(\cdot)\in\mathbb{R}} \frac{1}{2}\,\mathrm{e}^{\mathrm{j}\varphi} + \frac{\partial J}{\partial s^*}\left(\frac{1}{2}\,\mathrm{e}^{-\mathrm{j}\varphi}\right)^* = \frac{\partial J}{\partial s^*}\,\mathrm{e}^{\mathrm{j}\varphi}, \tag{4.16}$$

where $\frac{\partial J}{\partial s^*}$ is real-valued, because it is the gradient of a real-valued number.

4.8 Phase factor

For the operation

$$s = g(z) = \mathrm{e}^{\mathrm{j}\varphi} = \frac{z}{|z|} \tag{4.17}$$

(sometimes called the generalization of the sign function to complex numbers) we want to demonstrate two derivations. One way over the real derivatives (left column) and one by using the chain rule with known derivatives (right column).

For the real-valued derivative we insert $z = x + \mathrm{j}y$ in the operation

$$s = \mathrm{e}^{\mathrm{j}\varphi} = \frac{x + \mathrm{j}y}{\sqrt{x^2 + y^2}}$$

Using the chain rules requires to introduce intermediate variables

$$s = \mathrm{e}^{\mathrm{j}\varphi} = \frac{z}{|z|} =: \frac{z_1}{|z_3|} =: \frac{z_1}{z_2}, \tag{4.18}$$

$$z_2 \in \mathbb{R}. \tag{4.19}$$

and calculate the derivative w.r.t. x

$$\frac{\partial s}{\partial x} = \frac{1\sqrt{x^2+y^2} - (x+\mathrm{j}y)\frac{x}{\sqrt{x^2+y^2}}}{x^2+y^2}$$

$$= \frac{|z| - x\frac{z}{|z|}}{|z|^2}.$$

The calculation w.r.t. y is analogue

$$\frac{\partial s}{\partial y} = \frac{\mathrm{j}\,|z| - y\frac{z}{|z|}}{|z|^2}.$$

Insert both in the definition (3.8)

$$\frac{\partial s}{\partial z^*} = \frac{1}{2}\left(\frac{|z| - x\frac{z}{|z|} + \mathrm{j}\cdot\mathrm{j}\,|z| - \mathrm{j}y\frac{z}{|z|}}{|z|^2}\right)$$

$$= \frac{-z}{2\,|z|^3}\,(x+\mathrm{j}y)$$

$$= \frac{-1}{2\,|z|}\,\mathrm{e}^{\mathrm{j}2\varphi},$$

$$\frac{\partial s}{\partial z} = \frac{1}{2}\left(\frac{|z| - x\frac{z}{|z|} - \mathrm{j}\cdot\mathrm{j}\,|z| + \mathrm{j}y\frac{z}{|z|}}{|z|^2}\right)$$

$$= \frac{1}{|z|} - \frac{z}{2\,|z|^3}\,(x-\mathrm{j}y)$$

$$= \frac{1}{|z|} - \frac{1}{2\,|z|} = \frac{1}{2\,|z|}$$

and (3.7) results in the solution:

$$\frac{\partial J}{\partial z^*} = \left(\frac{\partial J}{\partial s^*}\right)^* \frac{-1}{2\,|z|}\,\mathrm{e}^{\mathrm{j}2\varphi} + \frac{\partial J}{\partial s^*}\left(\frac{1}{2\,|z|}\right)^*.$$

From the division we know

$$\frac{\partial J}{\partial z_1^*} = \frac{\partial J}{\partial s^*}\frac{1}{z_2^*},$$

$$\frac{\partial J}{\partial z_2^*} = \Re\left\{\frac{\partial J}{\partial s^*}\frac{-z_1^*}{(z_2^*)^2}\right\},$$

because z_2 is real-valued $\frac{\partial J}{\partial z_2^*}$ has to be forced to be real-valued (Section 3.3), too. Next use the absolute value

$$\frac{\partial J}{\partial z_3^*} = \frac{\partial J}{\partial z_2^*}\,\mathrm{e}^{\mathrm{j}\varphi}$$

$$= \Re\left\{\frac{\partial J}{\partial s^*}\frac{-z_1^*}{(z_2^*)^2}\right\}\mathrm{e}^{\mathrm{j}\varphi}$$

and combine the gradients

$$\frac{\partial J}{\partial z^*} = \frac{\partial J}{\partial z_1^*} + \frac{\partial J}{\partial z_3^*}$$

$$= \frac{\partial J}{\partial s^*}\frac{1}{z_2^*} + \Re\left\{\frac{\partial J}{\partial s^*}\frac{-z_1^*}{(z_2^*)^2}\right\}\mathrm{e}^{\mathrm{j}\varphi}$$

$$= \frac{\partial J}{\partial s^*}\frac{1}{|z|} + \Re\left\{\frac{\partial J}{\partial s^*}\frac{-z^*}{|z|^2}\right\}\mathrm{e}^{\mathrm{j}\varphi}$$

$$= \frac{\partial J}{\partial s^*}\frac{1}{|z|} - \Re\left\{\frac{\partial J}{\partial s^*}\frac{1}{z}\right\}\mathrm{e}^{\mathrm{j}\varphi}. \quad (4.20)$$

For comparison with the real-valued solution reformulate the equation

$$\frac{\partial J}{\partial z^*} = \frac{\partial J}{\partial s^*}\frac{1}{|z|} - \frac{1}{2}\left(\frac{\partial J}{\partial s^*}\frac{1}{z} + \left(\frac{\partial J}{\partial s^*}\frac{1}{z}\right)^*\right)\mathrm{e}^{\mathrm{j}\varphi}$$

$$= \frac{1}{2}\frac{\partial J}{\partial s^*}\frac{1}{|z|} - \frac{1}{2}\left(\frac{\partial J}{\partial s^*}\right)^*\frac{\mathrm{e}^{\mathrm{j}2\varphi}}{|z|}.$$

As expected both derivations results in the same solution. For the real-valued way it is easy to derive the gradient, because all derivation rules are well known and the solution must only be inserted in the definitions. The complex chain rules has to be used more carefully, because when an intermediate variable is real-valued the gradient must be forced to be real-valued, too.

4.9 Real/Imaginary part

The calculations for the real and imaginary part are similar.

The starting equation for the real part is

$$s = g(z) = \Re\{z\} = \Re\{x + \mathrm{j}y\} = x \tag{4.21}$$

The derivatives w.r.t. x and y are

$$\frac{\partial s}{\partial x} = \frac{\partial x}{\partial x} = 1 \quad \text{and} \quad \frac{\partial s}{\partial y} = \frac{\partial x}{\partial y} = 0.$$

Inserting in (3.8)

$$\frac{\partial s}{\partial z^*} = \frac{1}{2}\left(\frac{\partial s}{\partial x} + \mathrm{j}\frac{\partial s}{\partial y}\right) = \frac{1}{2},$$

$$\frac{\partial s}{\partial z} = \frac{1}{2}\left(\frac{\partial s}{\partial x} - \mathrm{j}\frac{\partial s}{\partial y}\right) = \frac{1}{2},$$

and (3.7)

$$\frac{\partial J}{\partial z^*} = \underbrace{\left(\frac{\partial J}{\partial s^*}\right)^*}_{(\cdot)\in\mathbb{R}}\frac{1}{2} + \frac{\partial J}{\partial s^*}\left(\frac{1}{2}\right)^*$$

$$= \frac{\partial J}{\partial s^*} \tag{4.22}$$

yields the gradient.

The starting equation for the imaginary part is

$$s = g(z) = \Im\{z\} = \Im\{x + \mathrm{j}y\} = y \tag{4.23}$$

The derivatives w.r.t. x and y are

$$\frac{\partial s}{\partial x} = \frac{\partial y}{\partial x} = 0 \quad \text{and} \quad \frac{\partial s}{\partial y} = \frac{\partial y}{\partial y} = 1.$$

Inserting in (3.8)

$$\frac{\partial s}{\partial z^*} = \frac{1}{2}\left(\frac{\partial s}{\partial x} + \mathrm{j}\frac{\partial s}{\partial y}\right) = \mathrm{j}\frac{1}{2},$$

$$\frac{\partial s}{\partial z} = \frac{1}{2}\left(\frac{\partial s}{\partial x} - \mathrm{j}\frac{\partial s}{\partial y}\right) = -\mathrm{j}\frac{1}{2},$$

and (3.7)

$$\frac{\partial J}{\partial z^*} = \underbrace{\left(\frac{\partial J}{\partial s^*}\right)^*}_{(\cdot)\in\mathbb{R}}\mathrm{j}\frac{1}{2} + \frac{\partial J}{\partial s^*}\left(-\mathrm{j}\frac{1}{2}\right)^*$$

$$= \mathrm{j}\frac{\partial J}{\partial s^*} \tag{4.24}$$

yields the gradient.

4.10 DFT/IDFT

We begin with the definition of the DFT

$$s_f = \sum_{n=0}^{N-1} z_n\, \mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}nf} \tag{4.25}$$

$$= \underset{n\to f}{\mathrm{DFT}}\{z_n\}$$

with z_n as the discrete time signal, s_f as transformed signal in the frequency domain and $n, f \in \{0, \dots, N\}$. Deriving w.r.t. z_n

We begin with the definition of the inverse discrete fourier transformation (IDFT)

$$s_n = \frac{1}{N}\sum_{f=0}^{N-1} z_f\, \mathrm{e}^{\mathrm{j}\frac{2\pi}{N}nf} \tag{4.27}$$

$$= \underset{f\to n}{\mathrm{IDFT}}\{z_f\}$$

with z_f as the discrete signal in the frequency domain, s_n as transformed time signal and $n, f \in \{0, \dots, N\}$. Deriving w.r.t.

and z_n^* z_f and z_f^*

$$\frac{\partial s_f}{\partial z_n} = \mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}nf} \quad \text{and} \quad \frac{\partial s_f}{\partial z_n^*} = 0 \qquad\qquad \frac{\partial s_n}{\partial z_f} = \frac{1}{N}\,\mathrm{e}^{\mathrm{j}\frac{2\pi}{N}nf} \quad \text{and} \quad \frac{\partial s_n}{\partial z_f^*} = 0$$

and inserting in (3.9)   and inserting in (3.9)

$$\frac{\partial J}{\partial z_n^*} = \sum_{f=0}^{N-1}\left(\left(\frac{\partial J}{\partial s_f^*}\right)^* 0 + \frac{\partial J}{\partial s_f^*}\left(\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}nf}\right)^*\right) \qquad \frac{\partial J}{\partial z_f^*} = \frac{1}{N}\sum_{n=0}^{N-1}\left(\left(\frac{\partial J}{\partial s_n^*}\right)^* 0 + \frac{\partial J}{\partial s_n^*}\left(\mathrm{e}^{\mathrm{j}\frac{2\pi}{N}nf}\right)^*\right)$$

$$= \sum_{f=0}^{N-1}\frac{\partial J}{\partial s_f^*}\,\mathrm{e}^{\mathrm{j}\frac{2\pi}{N}nf} \qquad\qquad\qquad = \frac{1}{N}\sum_{n=0}^{N-1}\frac{\partial J}{\partial s_n^*}\,\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}nf}$$

$$= N\cdot\underset{f\to n}{\mathrm{IDFT}}\left\{\frac{\partial J}{\partial s_f^*}\right\} \qquad (4.26) \qquad = \frac{1}{N}\cdot\underset{n\to f}{\mathrm{DFT}}\left\{\frac{\partial J}{\partial s_n^*}\right\} \qquad (4.28)$$

leads to the gradient.   leads to the gradient.

4.11 RDFT/IRDFT

The size of the DFT for real input is assumed to be even. We begin with the definition of the discrete fourier transformation for real input (RDFT)

$$s_f = \sum_{n=0}^{N-1} z_n\,\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}nf} = \underset{n\to f}{\mathrm{RDFT}}\{z_n\} \qquad (4.29)$$

with z_n as the discrete time signal, s_f as transformed signal in the frequency domain and $n\in\{0,\dots,N\}, f\in\{0,\dots,\frac{N}{2}\}$. Deriving w.r.t. z_n and z_n^*

$$\frac{\partial s_f}{\partial z_n} = \mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}nf} \quad \text{and} \quad \frac{\partial s_f}{\partial z_n^*} = 0$$

and inserting in (3.9) leads to

$$\frac{\partial J}{\partial z_n^*} = \sum_{f=0}^{N/2}\left(\left(\frac{\partial J}{\partial s_f^*}\right)^* 0 + \frac{\partial J}{\partial s_f^*}\left(\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}nf}\right)^*\right)$$

$$= \sum_{f=0}^{N/2}\frac{\partial J}{\partial s_f^*}\,\mathrm{e}^{\mathrm{j}\frac{2\pi}{N}nf}\,.$$

Note, that the sum over f is limited by $N/2$. Since the input is real-valued the gradient has to be real-valued, too. This leads to

$$
\begin{aligned}
\frac{\partial J}{\partial z_n^*} &= \Re\left\{\sum_{f=0}^{N/2} \frac{\partial J}{\partial s_f^*}\, e^{j\frac{2\pi}{N}nf}\right\} \\
&= \sum_{f\in\{0,\frac{N}{2}\}} \Re\left\{\frac{\partial J}{\partial s_f^*}\right\} \underbrace{e^{j\frac{2\pi}{N}nf}}_{\in\mathbb{R}} + \sum_{f=1}^{N/2-1} \Re\left\{\frac{\partial J}{\partial s_f^*}\, e^{j\frac{2\pi}{N}nf}\right\} \\
&= \sum_{f\in\{0,\frac{N}{2}\}} \Re\left\{\frac{\partial J}{\partial s_f^*}\right\} \underbrace{e^{j\frac{2\pi}{N}nf}}_{\in\mathbb{R}} + \sum_{f=1}^{N/2-1} \frac{1}{2}\left(\frac{\partial J}{\partial s_f^*}\, e^{j\frac{2\pi}{N}nf} + \left(\frac{\partial J}{\partial s_f^*}\right)^* e^{-j\frac{2\pi}{N}nf}\right).
\end{aligned}
$$
$$(4.30)$$

Using the definition of the IRDFT (4.33) and

$$
\frac{\widetilde{\partial J}}{\partial s_f^*} = \begin{cases} \Re\left\{\frac{\partial J}{\partial s_f^*}\right\}, & \text{for } f\in\left\{0,\frac{N}{2}\right\}, \\ \frac{1}{2}\frac{\partial J}{\partial s_f^*}, & \text{for } f\in\left[1,\ldots,\frac{N}{2}-1\right] \end{cases}
$$
$$(4.31)$$

the gradient can be calculated with

$$
\frac{\partial J}{\partial z_n^*} = N\cdot\underset{f\to n}{\mathrm{IRDFT}}\left\{\frac{\widetilde{\partial J}}{\partial s_f^*}\right\}.
$$
$$(4.32)$$

For the derivation of the inverse discrete fourier transformation for real input (IRDFT) we start with its definition

$$
s_n = \frac{1}{N}\sum_{f=0}^{N/2} z_f\, e^{j\frac{2\pi}{N}nf} + \frac{1}{N}\sum_{f=1}^{N/2-1} z_f^*\, e^{-j\frac{2\pi}{N}nf} = \underset{f\to n}{\mathrm{IRDFT}}\{z_f\}
$$
$$(4.33)$$

with z_f as the discrete signal in the frequency domain, s_n as transformed time signal and $n\in\{0,\ldots,N\}, f\in\{0,\ldots,\frac{N}{2}\}$. Deriving w.r.t. z_f and z_f^*

$$
\frac{\partial s_f}{\partial z_f} = \frac{1}{N}\, e^{j\frac{2\pi}{N}nf} \quad\text{and}\quad \frac{\partial s_f}{\partial z_f^*} = \frac{1}{N}\, e^{-j\frac{2\pi}{N}nf}
$$

and inserting in (3.9) leads to

$$
\begin{aligned}
\frac{\partial J}{\partial z_0^*} &= \frac{1}{N}\sum_{n=1}^{N/2-1} \frac{\partial J}{\partial s_n^*}1, \\
\frac{\partial J}{\partial z_{N/2}^*} &= \frac{1}{N}\sum_{n=1}^{N/2-1} \frac{\partial J}{\partial s_n^*}\, e^{j\pi n}, \\
\frac{\partial J}{\partial z_f^*} &= \frac{1}{N}\sum_{n=1}^{N/2-1} \left(\left(\frac{\partial J}{\partial s_n^*}\right)^* e^{-j\frac{2\pi}{N}nf} + \frac{\partial J}{\partial s_n^*}\left(e^{j\frac{2\pi}{N}nf}\right)^*\right) \\
&= \frac{1}{N}\sum_{n=1}^{N/2-1} 2\frac{\partial J}{\partial s_n^*}\, e^{-j\frac{2\pi}{N}nf} \quad \forall z_f, f\in\{1,\ldots,\frac{N}{2}-1\}.
\end{aligned}
$$

The last equation holds because the gradient of the objective function w.r.t. the output of the IRDFT s_n is real-valued since the output has to be real-valued, too. With the RDFT

$$\widetilde{\frac{\partial J}{\partial z_f^*}} = \frac{1}{N} \cdot \underset{n \to f}{\mathrm{RDFT}} \left\{ \frac{\partial J}{\partial s_n^*} \right\} \tag{4.34}$$

the gradient of the IRDFT can be obtained by

$$\frac{\partial J}{\partial z_f^*} = \begin{cases} \widetilde{\frac{\partial J}{\partial z_f^*}}, & \text{for } f \in \left\{ 0, \frac{N}{2} \right\}, \\ 2\widetilde{\frac{\partial J}{\partial z_f^*}}, & \text{for } f \in \left[1, \dots, \frac{N}{2} - 1 \right]. \end{cases} \tag{4.35}$$

4.12 Vector normalization

We define the vector normalization such that the euclidean norm of the output vector is one

$$\mathbf{s} = \frac{\mathbf{z}}{\sqrt{\mathbf{z}^H \mathbf{z}}}. \tag{4.36}$$

The way to determine the gradient with the approach from Section 3.5.2 is more complicated than the approach from Section 3.5.1. Therefore we start with the elementwise formulation

$$s_n = \frac{z_n}{\sqrt{\sum_m z_m^* z_m}},$$

calculate the derivative w.r.t. z_m^*

$$\frac{\partial s_n}{\partial z_m^*} = -\frac{z_n}{\sum_{\check{m}} z_{\check{m}}^* z_{\check{m}}} \frac{1}{2\sqrt{\sum_{\check{m}} z_{\check{m}}^* z_{\check{m}}}} z_m = \frac{-s_n z_m}{2\sum_{\check{m}} z_{\check{m}}^* z_{\check{m}}}$$

and insert it in (3.12)

$$\frac{\partial \mathbf{s}}{\partial \mathbf{z}^*} = -\frac{\mathbf{s}\mathbf{z}^T}{2\mathbf{z}^H \mathbf{z}}.$$

The calculation for $\frac{\partial \mathbf{s}}{\partial \mathbf{z}}$ is analogue

$$\frac{\partial s_n}{\partial z_m} = -\frac{z_n}{\sum_{\check{m}} z_{\check{m}}^* z_{\check{m}}} \frac{1}{2\sqrt{\sum_{\check{m}} z_{\check{m}}^* z_{\check{m}}}} z_m^* + \frac{\delta_{nm}}{\sqrt{\sum_{\check{m}} z_{\check{m}}^* z_{\check{m}}}},$$

$$\frac{\partial \mathbf{s}}{\partial \mathbf{z}} = \frac{\mathbf{I}}{\sqrt{\mathbf{z}^H \mathbf{z}}} - \frac{\mathbf{s}\mathbf{z}^H}{2\mathbf{z}^H \mathbf{z}},$$

where δ_{nm} is the Kronecker delta

$$\delta_{nm} = \begin{cases} 1, & \text{if } m = n, \\ 0, & \text{if } m \neq n. \end{cases} \tag{4.37}$$

Inserting this intermediate result in (3.11)

$$
\begin{aligned}
\frac{\partial J}{\partial \mathbf{z}^*} &= \left(\left(\frac{\partial J}{\partial \mathbf{s}^*} \right)^{\mathrm{H}} \frac{\partial \mathbf{s}}{\partial \mathbf{z}^*} + \left(\frac{\partial J}{\partial \mathbf{s}^*} \right)^{\mathrm{T}} \left(\frac{\partial \mathbf{s}}{\partial \mathbf{z}} \right)^* \right)^{\mathrm{T}} \\
&= \left(-\left(\frac{\partial J}{\partial \mathbf{s}^*} \right)^{\mathrm{H}} \frac{\mathbf{s}\mathbf{z}^{\mathrm{T}}}{2\mathbf{z}^{\mathrm{H}}\mathbf{z}} + \left(\frac{\partial J}{\partial \mathbf{s}^*} \right)^{\mathrm{T}} \left(\frac{\mathbf{I}}{\sqrt{\mathbf{z}^{\mathrm{H}}\mathbf{z}}} - \frac{\mathbf{s}\mathbf{z}^{\mathrm{H}}}{2\mathbf{z}^{\mathrm{H}}\mathbf{z}} \right)^* \right)^{\mathrm{T}} \\
&= -\frac{\mathbf{z}\mathbf{s}^{\mathrm{T}}}{2\mathbf{z}^{\mathrm{H}}\mathbf{z}} \left(\frac{\partial J}{\partial \mathbf{s}^*} \right)^* + \left(\frac{\mathbf{I}}{\sqrt{\mathbf{z}^{\mathrm{H}}\mathbf{z}}} - \frac{\mathbf{z}^*\mathbf{s}^{\mathrm{T}}}{2\mathbf{z}^{\mathrm{H}}\mathbf{z}} \right)^* \frac{\partial J}{\partial \mathbf{s}^*} \\
&= \frac{1}{\sqrt{\mathbf{z}^{\mathrm{H}}\mathbf{z}}} \frac{\partial J}{\partial \mathbf{s}^*} - \frac{\mathbf{z}\mathbf{s}^{\mathrm{T}}}{2\mathbf{z}^{\mathrm{H}}\mathbf{z}} \left(\frac{\partial J}{\partial \mathbf{s}^*} \right)^* - \frac{\mathbf{z}\mathbf{s}^{\mathrm{H}}}{2\mathbf{z}^{\mathrm{H}}\mathbf{z}} \frac{\partial J}{\partial \mathbf{s}^*} \\
&= \frac{1}{\sqrt{\mathbf{z}^{\mathrm{H}}\mathbf{z}}} \frac{\partial J}{\partial \mathbf{s}^*} - \frac{\mathbf{z}}{2\mathbf{z}^{\mathrm{H}}\mathbf{z}} \left(\mathbf{s}^{\mathrm{T}} \left(\frac{\partial J}{\partial \mathbf{s}^*} \right)^* + \mathbf{s}^{\mathrm{H}} \frac{\partial J}{\partial \mathbf{s}^*} \right) \\
&= \frac{1}{\sqrt{\mathbf{z}^{\mathrm{H}}\mathbf{z}}} \frac{\partial J}{\partial \mathbf{s}^*} - \frac{\mathbf{z}}{\mathbf{z}^{\mathrm{H}}\mathbf{z}} \Re \left\{ \mathbf{s}^{\mathrm{H}} \frac{\partial J}{\partial \mathbf{s}^*} \right\} \\
&= \frac{1}{\sqrt{\mathbf{z}^{\mathrm{H}}\mathbf{z}}} \frac{\partial J}{\partial \mathbf{s}^*} - \frac{\mathbf{z}}{\sqrt{\mathbf{z}^{\mathrm{H}}\mathbf{z}}^3} \Re \left\{ \mathbf{z}^{\mathrm{H}} \frac{\partial J}{\partial \mathbf{s}^*} \right\} \\
&= \frac{\frac{\partial J}{\partial \mathbf{s}^*} - \frac{\mathbf{z}}{\mathbf{z}^{\mathrm{H}}\mathbf{z}} \Re \left\{ \mathbf{z}^{\mathrm{H}} \frac{\partial J}{\partial \mathbf{s}^*} \right\}}{\sqrt{\mathbf{z}^{\mathrm{H}}\mathbf{z}}}
\end{aligned}
\tag{4.38}
$$

yields the gradient.

4.13 Matrix multiplication

For the matrix multiplication

$$
\mathbf{C} = \mathbf{g}(\mathbf{A}, \mathbf{B}) = \mathbf{A}\mathbf{B}
\tag{4.39}
$$

we want to show two ways for the calculation of the gradient. In the left column we show the way like in the previous sections (Section 3.5.1) and use the elementwise notation

$$
\begin{aligned}
c_{nm} &= g(a_{n1}, \ldots, a_{nK}, b_{1m}, \ldots, b_{Km}) \\
&= \sum_k a_{nk} b_{km}.
\end{aligned}
$$

Note, that here the variable names have changed. In the previous sections the inputs/outputs are z_1, z_2 and s and now they are $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$. This change is done, because the algorithm in Section 3.5.2 requires a pseudo input variable, which we call $\zeta \in \{z, z^*\}$. This algorithm is used in the right column.

We start with elementwise notation

$$c_{n,m} = \sum_k a_{nk} b_{k,m},$$

calculate all necessary derivatives

$$\frac{\partial c_{n,m}}{\partial a_{n,k}} = b_{k,m},$$

$$\frac{\partial c_{n,m}}{\partial a_{n,k}^*} = 0,$$

$$\frac{\partial c_{n,m}}{\partial b_{k,m}} = a_{n,k},$$

$$\frac{\partial c_{n,m}}{\partial b_{k,m}^*} = 0.$$

and inserting them in (3.9)

$$\frac{\partial J}{\partial a_{n,k}^*} = \left(\frac{\partial J}{\partial c_{n,m}^*} \right)^* 0$$

$$+ \frac{\partial J}{\partial c_{n,m}^*} b_{k,m}^*$$

$$\frac{\partial J}{\partial \mathbf{A}^*} = \frac{\partial J}{\partial \mathbf{C}^*} \mathbf{B}^{\mathrm{H}} \quad (4.40)$$

$$\frac{\partial J}{\partial b_{k,m}^*} = \left(\frac{\partial J}{\partial c_{n,m}^*} \right)^* 0$$

$$+ \frac{\partial J}{\partial c_{n,m}^*} a_{nk}^*$$

$$\frac{\partial J}{\partial \mathbf{B}^*} = \mathbf{A}^{\mathrm{H}} \frac{\partial J}{\partial \mathbf{C}^*} \quad (4.41)$$

yields the result.

Start with calculating the derivative[1] of both sides of $\mathbf{C} = \mathbf{AB}$ w.r.t. $\zeta \in \{z, z^*\}$. In section A.1 it is shown that the product rule for real matrices holds also in the complex case w.r.t. z and z^*, which gives

$$\frac{\partial \mathbf{C}}{\partial \zeta} = \frac{\partial \mathbf{A}}{\partial \zeta} \mathbf{B} + \mathbf{A} \frac{\partial \mathbf{B}}{\partial \zeta}.$$

Insert this in (3.18)

$$\frac{\partial J}{\partial z^*} = \mathrm{Tr}\left(\left(\frac{\partial J}{\partial \mathbf{C}^*} \right)^{\mathrm{H}} \frac{\partial \mathbf{C}}{\partial z^*} \right) + \dots$$

$$= \mathrm{Tr}\left(\left(\frac{\partial J}{\partial \mathbf{C}^*} \right)^{\mathrm{H}} \left(\frac{\partial \mathbf{A}}{\partial z^*} \mathbf{B} + \mathbf{A} \frac{\partial \mathbf{B}}{\partial z^*} \right) \right) + \dots$$

$$= \mathrm{Tr}\left(\left(\frac{\partial J}{\partial \mathbf{C}^*} \mathbf{B}^{\mathrm{H}} \right)^{\mathrm{H}} \frac{\partial \mathbf{A}}{\partial z^*} + \left(\mathbf{A}^{\mathrm{H}} \frac{\partial J}{\partial \mathbf{C}^*} \right)^{\mathrm{H}} \frac{\partial \mathbf{B}}{\partial z^*} \right) + \dots$$

The dots represent the part of (3.18), which is redundant in this example. Comparing the coefficients with (3.20)

$$\frac{\partial J}{\partial \mathbf{A}^*} = \frac{\partial J}{\partial \mathbf{C}^*} \mathbf{B}^{\mathrm{H}} \qquad (4.42)$$

$$\frac{\partial J}{\partial \mathbf{B}^*} = \mathbf{A}^{\mathrm{H}} \frac{\partial J}{\partial \mathbf{C}^*} \qquad (4.43)$$

yields the results.

---

[1]In Appendix A.2 it is shown, that it is not necessary to distinguish between z and z^* in this case.

As expected both ways lead to the same solution.

4.14 Matrix inverse

For the matrix inverse

$$\mathbf{C} = \mathbf{g}(\mathbf{A}) = \mathbf{A}^{-1} \tag{4.44}$$

we assume the inverse exists. An implicit formulation is

$$\mathbf{AC} = \mathbf{I}$$

and the derivation w.r.t. $\zeta \in \{z, z^*\}$ leads to

$$\frac{\partial \mathbf{A}}{\partial \zeta}\mathbf{C} + \mathbf{A}\frac{\partial \mathbf{C}}{\partial \zeta} = \mathbf{0},$$

$$\frac{\partial \mathbf{C}}{\partial \zeta} = -\mathbf{A}^{-1}\frac{\partial \mathbf{A}}{\partial \zeta}\mathbf{C}.$$

Inserting these equations in (3.18)

$$\frac{\partial J}{\partial z^*} = \mathrm{Tr}\left(\left(\frac{\partial J}{\partial \mathbf{C}^*}\right)^{\mathrm{H}}\frac{\partial \mathbf{C}}{\partial z^*}\right) + \dots$$

$$= \mathrm{Tr}\left(\left(\frac{\partial J}{\partial \mathbf{C}^*}\right)^{\mathrm{H}}\left(-\mathbf{A}^{-1}\frac{\partial \mathbf{A}}{\partial z^*}\mathbf{C}\right)\right) + \dots$$

$$= \mathrm{Tr}\left(-\mathbf{C}\left(\frac{\partial J}{\partial \mathbf{C}^*}\right)^{\mathrm{H}}\mathbf{A}^{-1}\frac{\partial \mathbf{A}}{\partial z^*}\right) + \dots$$

$$= \mathrm{Tr}\left(\left(-\mathbf{A}^{-\mathrm{H}}\frac{\partial J}{\partial \mathbf{C}^*}\mathbf{C}^{\mathrm{H}}\right)^{\mathrm{H}}\frac{\partial \mathbf{A}}{\partial z^*}\right) + \dots$$

$$= \mathrm{Tr}\left(\left(-\mathbf{C}^{\mathrm{H}}\frac{\partial J}{\partial \mathbf{C}^*}\mathbf{C}^{\mathrm{H}}\right)^{\mathrm{H}}\frac{\partial \mathbf{A}}{\partial z^*}\right) + \dots$$

and comparing the coefficients with (3.20)

$$\frac{\partial J}{\partial \mathbf{A}^*} = -\mathbf{C}^{\mathrm{H}}\frac{\partial J}{\partial \mathbf{C}^*}\mathbf{C}^{\mathrm{H}} \tag{4.45}$$

yields the gradient.

4.15 Matrix inverse product

The matrix inverse product

$$\mathbf{C} = \mathbf{g}(\mathbf{A}, \mathbf{B}) = \mathbf{A}^{-1}\mathbf{B} \tag{4.46}$$

is also called `solve` in many numerical implementations, because it solves the equation

$$\mathbf{A}\mathbf{C} = \mathbf{B}.$$

The derivative of this equation w.r.t. to $\zeta \in \{z, z^*\}$ is

$$\frac{\partial \mathbf{A}}{\partial \zeta}\mathbf{C} + \mathbf{A}\frac{\partial \mathbf{C}}{\partial \zeta} = \frac{\partial \mathbf{B}}{\partial \zeta},$$

$$\frac{\partial \mathbf{C}}{\partial \zeta} = \mathbf{A}^{-1}\frac{\partial \mathbf{B}}{\partial \zeta} - \mathbf{A}^{-1}\frac{\partial \mathbf{A}}{\partial \zeta}\mathbf{C}.$$

Now the same principle as in the previous section can be used. Inserting in (3.18)

$$
\begin{aligned}
\frac{\partial J}{\partial z^*} &= \mathrm{Tr}\left(\left(\frac{\partial J}{\partial \mathbf{C}^*}\right)^{\mathrm{H}} \frac{\partial \mathbf{C}}{\partial z^*}\right) + \dots \\
&= \mathrm{Tr}\left(\left(\frac{\partial J}{\partial \mathbf{C}^*}\right)^{\mathrm{H}} \mathbf{A}^{-1}\left(\frac{\partial \mathbf{B}}{\partial z^*} - \frac{\partial \mathbf{A}}{\partial z^*}\mathbf{C}\right)\right) + \dots \\
&= \mathrm{Tr}\left(\left(\mathbf{A}^{-\mathrm{H}}\frac{\partial J}{\partial \mathbf{C}^*}\right)^{\mathrm{H}} \frac{\partial \mathbf{B}}{\partial z^*} + \left(-\mathbf{A}^{-\mathrm{H}}\frac{\partial J}{\partial \mathbf{C}^*}\mathbf{C}^{\mathrm{H}}\right)^{\mathrm{H}} \frac{\partial \mathbf{A}}{\partial z^*}\right) + \dots
\end{aligned}
$$

and comparing with (3.20)

$$
\frac{\partial J}{\partial \mathbf{B}^*} = \mathbf{A}^{-\mathrm{H}}\frac{\partial J}{\partial \mathbf{C}^*} \tag{4.47}
$$

$$
\frac{\partial J}{\partial \mathbf{A}^*} = -\mathbf{A}^{-\mathrm{H}}\frac{\partial J}{\partial \mathbf{C}^*}\mathbf{C}^{\mathrm{H}} = -\frac{\partial J}{\partial \mathbf{B}^*}\mathbf{C}^{\mathrm{H}} \tag{4.48}
$$

yields the result.

For the second matrix inverse product

$$
\mathbf{C} = \mathbf{g}(\mathbf{A},\mathbf{B}) = \mathbf{A}\mathbf{B}^{-1} \tag{4.49}
$$

the calculation is analogue. Start with implicit formulation

$$
\mathbf{C}\mathbf{B} = \mathbf{A},
$$

calculate the derivative

$$
\frac{\partial \mathbf{C}}{\partial \zeta}\mathbf{B} + \mathbf{C}\frac{\partial \mathbf{B}}{\partial \zeta} = \frac{\partial \mathbf{A}}{\partial \zeta},
$$

$$
\frac{\partial \mathbf{C}}{\partial \zeta} = \frac{\partial A}{\partial \zeta}\mathbf{B}^{-1} - \mathbf{C}\frac{\partial \mathbf{B}}{\partial \zeta}\mathbf{B}^{-1},
$$

insert it in (3.18)

$$
\begin{aligned}
\frac{\partial J}{\partial z^*} &= \mathrm{Tr}\left(\left(\frac{\partial J}{\partial \mathbf{C}^*}\right)^{\mathrm{H}} \frac{\partial \mathbf{C}}{\partial z^*}\right) + \dots \\
&= \mathrm{Tr}\left(\left(\frac{\partial J}{\partial \mathbf{C}^*}\right)^{\mathrm{H}} \left(\frac{\partial A}{\partial z^*} - \mathbf{C}\frac{\partial \mathbf{B}}{\partial z^*}\right)\mathbf{B}^{-1}\right) + \dots \\
&= \mathrm{Tr}\left(\left(\frac{\partial J}{\partial \mathbf{C}^*}\mathbf{B}^{-\mathrm{H}}\right)^{\mathrm{H}} \frac{\partial \mathbf{A}}{\partial z^*} + \left(-\mathbf{C}^{\mathrm{H}}\frac{\partial J}{\partial \mathbf{C}^*}\mathbf{B}^{-\mathrm{H}}\right)^{\mathrm{H}} \frac{\partial \mathbf{B}}{\partial z^*}\right) + \dots
\end{aligned}
$$

and compare with (3.20)

$$
\frac{\partial J}{\partial \mathbf{A}^*} = \frac{\partial J}{\partial \mathbf{C}^*}\mathbf{B}^{-\mathrm{H}}, \tag{4.50}
$$

$$
\frac{\partial J}{\partial \mathbf{B}^*} = -\mathbf{C}^{\mathrm{H}}\frac{\partial J}{\partial \mathbf{C}^*}\mathbf{B}^{-\mathrm{H}} = -\mathbf{C}^{\mathrm{H}}\frac{\partial J}{\partial \mathbf{A}^*} \tag{4.51}
$$

which yields the result again.

4.16 Cholesky decomposition

The derivation for the real-valued Cholesky decomposition is described in [Mur16, Chapter 3] and the following complex-valued calculation is similar to it. The Cholesky decomposition is denoted as

$$\mathbf{L} = \mathbf{g}(\mathbf{A}) = \mathrm{cholesky}\,(\mathbf{A}) \tag{4.52}$$

where $\mathbf{A} = \mathbf{L}\mathbf{L}^{\mathrm{H}}$ and $\mathbf{L}$ is a lower triangular matrix (in the following $L \in \blacktriangle$ and $U \in \blacktriangledown$ indicate a lower and upper triangular matrix, respectively). Also we assume, that $\mathbf{A}$ is a hermitian matrix ($\mathbf{A} = \mathbf{A}^{\mathrm{H}}$). Therefore the operation $\mathbf{A} \overset{!}{=} \frac{1}{2}\left(\mathbf{B} + \mathbf{B}^{\mathrm{H}}\right)$ where $\mathbf{B} = \mathbf{A}$ holds. The gradient of $\mathbf{B}$ is then $\frac{\partial J}{\partial \mathbf{B}^{*}} = \frac{1}{2}\left(\frac{\partial J}{\partial \mathbf{A}^{*}} + \left(\frac{\partial J}{\partial \mathbf{A}^{*}}\right)^{\mathrm{H}}\right)$ (see Sections 4.2 and 4.3).

Some properties of triangular matrices and the Cholesky decomposition:

Prop.1) If $\mathbf{L} \in \blacktriangle$ then $\mathbf{L}^{-1} \in \blacktriangle$.

Prop.2) If $\mathbf{L}_1 \in \blacktriangle$ and $\mathbf{L}_2 \in \blacktriangle$ then $\mathbf{L}_1 \mathbf{L}_2 \in \blacktriangle$.

Prop.3) If $\mathbf{L} \in \blacktriangle$ then $[\mathbf{L}^{-1}]_{i,i} = (L_{i,i})^{-1}$ ($[\cdot]_{i,j}$ takes element from row i and column j).

Prop.4) If $\mathbf{L}_1, \mathbf{L}_2, \mathbf{L}_3 \in \blacktriangle$ and $\mathbf{L}_3 = \mathbf{L}_1 \mathbf{L}_2$ then $[\mathbf{L}_3]_{i,i} = [\mathbf{L}_1]_{i,i} [\mathbf{L}_2]_{i,i}$.

Prop.5) diagonal entries of $\mathbf{L} = \mathbf{g}(\mathbf{A})$ are real and positive (from previous follows: diagonal entries of $\mathbf{L}^{-1}$ are also real and positive).

Prop.6) If $\mathbf{L}_1, \mathbf{L}_2 \in \blacktriangle$, $A_2 = \mathbf{L}_2 \mathbf{L}_2^{\mathrm{H}}$ and $\mathbf{A}_3 = \mathbf{L}_1 \mathbf{A}_2 \mathbf{L}_1^{\mathrm{H}}$ then $\mathbf{A}_3 = \mathbf{L}_1 \mathbf{L}_2 \left(\mathbf{L}_1 \mathbf{L}_2\right)^{\mathrm{H}}$ is a hermitian matrix ($\mathbf{A}_3 = \mathbf{A}_3^{\mathrm{H}}$).

Prop.7) If $\mathbf{L} \in \blacktriangle$, $L_{i,i} \in \mathbb{R}$, $\mathbf{C} = \mathbf{C}^{\mathrm{H}}$ and $\mathbf{C} = \mathbf{L} + \mathbf{L}^{\mathrm{H}}$ then $\left(\triangle - \frac{1}{2}\mathbf{I}\right) \circ \mathbf{C} = \mathbf{L}$, where $\triangle$ is a lower triangular matrix filled with ones on the diagonal and below and $\circ$ denotes the Hadamard product (also known as elementwise multiplication).

We start with the equation $\mathbf{L} = \mathbf{g}(\mathbf{A})$ and calculate the derivative w.r.t. ζ

$$\frac{\partial \mathbf{A}}{\partial \zeta} = \frac{\partial \mathbf{L}}{\partial \zeta}\mathbf{L}^{\mathrm{H}} + \mathbf{L}\frac{\partial \mathbf{L}^{\mathrm{H}}}{\partial \zeta}.$$

After reformulation we obtain:

$$\mathbf{L}^{-1}\frac{\partial \mathbf{A}}{\partial \zeta}\mathbf{L}^{-\mathrm{H}} = \mathbf{L}^{-1}\frac{\partial \mathbf{L}}{\partial \zeta} + \frac{\partial \mathbf{L}^{\mathrm{H}}}{\partial \zeta}\mathbf{L}^{-\mathrm{H}},$$

$$\mathbf{L}^{-1}\frac{\partial \mathbf{A}}{\partial \zeta}\mathbf{L}^{-\mathrm{H}} = \underbrace{\mathbf{L}^{-1}\frac{\partial \mathbf{L}}{\partial \zeta}}_{(\cdot)\in\blacktriangle} + \underbrace{\left(\mathbf{L}^{-1}\frac{\partial \mathbf{L}}{\partial \zeta}\right)^{\mathrm{H}}}_{(\cdot)\in\blacktriangledown}.$$

Assuming that the imaginary part of the diagonal of $\mathbf{L}^{-1}\frac{\partial \mathbf{L}}{\partial \zeta}$ is zero[1], we can reformulate the equation:

$$\left(\triangle - \frac{1}{2}\mathbf{I}\right) \circ \left(\mathbf{L}^{-1}\frac{\partial \mathbf{A}}{\partial \zeta}\mathbf{L}^{-H}\right) = \mathbf{L}^{-1}\frac{\partial \mathbf{L}}{\partial \zeta},$$

$$\frac{\partial \mathbf{L}}{\partial \zeta} = \mathbf{L}\left(\left(\triangle - \frac{1}{2}\mathbf{I}\right) \circ \left(\mathbf{L}^{-1}\frac{\partial \mathbf{A}}{\partial \zeta}\mathbf{L}^{-H}\right)\right).$$

Inserting in (3.18)

$$\frac{\partial J}{\partial z^*} = \text{Tr}\left(\left(\frac{\partial J}{\partial \mathbf{L}^*}\right)^{H}\frac{\partial \mathbf{L}}{\partial z^*}\right) + \dots$$

$$= \text{Tr}\left(\left(\frac{\partial J}{\partial \mathbf{L}^*}\right)^{H}\mathbf{L}\left(\left(\triangle - \frac{1}{2}\mathbf{I}\right) \circ \left(\mathbf{L}^{-1}\frac{\partial \mathbf{A}}{\partial \zeta}\mathbf{L}^{-H}\right)\right)\right) + \dots$$

and using $\text{Tr}\left(\mathbf{A}\left(\mathbf{B} \circ \mathbf{C}\right)\right) = \text{Tr}\left(\left(\mathbf{A} \circ \mathbf{B}^{T}\right)\mathbf{C}\right)$ [Mil07, Theorem 2.6] results in:

$$\frac{\partial J}{\partial z^*} = \text{Tr}\left(\left(\left(\left(\frac{\partial J}{\partial \mathbf{L}^*}\right)^{H}\mathbf{L}\right) \circ \left(\triangle - \frac{1}{2}\mathbf{I}\right)^{T}\right)\mathbf{L}^{-1}\frac{\partial \mathbf{A}}{\partial \zeta}\mathbf{L}^{-H}\right) + \dots$$

$$= \text{Tr}\left(\mathbf{L}^{-H}\left(\left(\left(\frac{\partial J}{\partial \mathbf{L}^*}\right)^{H}\mathbf{L}\right) \circ \left(\triangle - \frac{1}{2}\mathbf{I}\right)^{T}\right)\mathbf{L}^{-1}\frac{\partial \mathbf{A}}{\partial \zeta}\right) + \dots$$

$$= \text{Tr}\left(\left(\mathbf{L}^{-H}\left(\left(\mathbf{L}^{H}\frac{\partial J}{\partial \mathbf{L}^*}\right) \circ \left(\triangle - \frac{1}{2}\mathbf{I}\right)\right)\mathbf{L}^{-1}\right)^{H}\frac{\partial \mathbf{A}}{\partial \zeta}\right) + \dots$$

With the comparison (3.20) we receive

$$\frac{\partial J}{\partial \mathbf{A}^*} = \mathbf{L}^{-H}\left(\left(\mathbf{L}^{H}\frac{\partial J}{\partial \mathbf{L}^*}\right) \circ \left(\triangle - \frac{1}{2}\mathbf{I}\right)\right)\mathbf{L}^{-1}. \tag{4.53}$$

Taking into account, that the input for the Cholesky decomposition must be a hermitian matrix, it is intuitive to ensure that the gradient is also a hermitian matrix (Note: in general $\frac{\partial J}{\partial \mathbf{A}^*}$ is not a hermitian matrix)

$$\frac{\partial J}{\partial \mathbf{B}^*} = \frac{1}{2}\left(\frac{\partial J}{\partial \mathbf{A}^*} + \left(\frac{\partial J}{\partial \mathbf{A}^*}\right)^{H}\right). \tag{4.54}$$

This force hermitian operation is debatable, because simulations have shown, that for a calculation, which produces a hermitian matrix, it makes no difference for the gradient if we take $\frac{\partial J}{\partial \mathbf{A}^*}$ or $\frac{\partial J}{\partial \mathbf{B}^*}$. On the contrary, there is an analytical interpretation, where $\mathbf{A}$ is a parameter, which needs an update $\mathbf{A}^{\text{new}} = \mathbf{A}^{\text{old}} - \mu\frac{\partial J}{\partial \mathbf{A}^*}$. Here the gradient must ensure that $\mathbf{A}$ remains hermitian. To ensure this, the gradient has to be hermitian. Therefore we use $\frac{\partial J}{\partial \mathbf{B}^*}$.

---

[1] The assumption that the imaginary part of the diagonal of $\mathbf{L}^{-1}\frac{\partial \mathbf{L}}{\partial \zeta}$ has to be zero comes from Prop.7. We have no analytical solution, that this is the case. But the final solution for the gradient survives a numerical test. We know: If and only if the diagonal from $\frac{\partial \mathbf{L}}{\partial \zeta}$ is real valued this is the case.

4.17 Eigendecomposition

To perform the Beamforming operation in a NN we need the derivate of the eigende-composition with complex-valued eigenvectors. This derivation is similar to the real valued version in [Gil08]. We start with the eigenvalue equation

$$\boldsymbol{\Phi}\mathbf{W} = \mathbf{W}\boldsymbol{\Lambda} \tag{4.55}$$

where $\boldsymbol{\Phi}$, $\mathbf{W}$ and $\boldsymbol{\Lambda}$ are a quadratic matrix, the eigenvector matrix and the diagonal eigenvalue matrix, respectively. At first we examine the forward mode AD. With the matrix product rule the derivation can be written as

$$\frac{\partial\boldsymbol{\Phi}}{\partial\zeta}\mathbf{W} + \boldsymbol{\Phi}\frac{\partial\mathbf{W}}{\partial\zeta} = \frac{\partial\mathbf{W}}{\partial\zeta}\boldsymbol{\Lambda} + \mathbf{W}\frac{\partial\boldsymbol{\Lambda}}{\partial\zeta}, \tag{4.56}$$

where $\zeta \in \{z, z^*\}$. Since the matrix of eigenvalues is a diagonal matrix we use the rule (proof in Appendix A.3)

$$\mathbf{C}\boldsymbol{\Lambda} - \boldsymbol{\Lambda}\mathbf{C} = \mathbf{E} \circ \mathbf{C}, \tag{4.57}$$

where $(A \circ B)_{ij} = A_{ij}B_{ij}$ denotes the Hadamard product and $E_{ij} = \lambda_j - \lambda_i$ are the entries of $\mathbf{E}$, which is a matrix of the differences of the eigenvalues [Gil08].

With the identity $\boldsymbol{\Phi} = \mathbf{W}\boldsymbol{\Lambda}\mathbf{W}^{-1}$, (4.56) premultiplied with $\mathbf{W}^{-1}$

$$\mathbf{W}^{-1}\frac{\partial\boldsymbol{\Phi}}{\partial\zeta}\mathbf{W} - \frac{\partial\boldsymbol{\Lambda}}{\partial\zeta} = \mathbf{W}^{-1}\frac{\partial\mathbf{W}}{\partial\zeta}\boldsymbol{\Lambda} - \boldsymbol{\Lambda}\mathbf{W}^{-1}\frac{\partial\mathbf{W}}{\partial\zeta} \tag{4.58}$$

and (4.57) we get

$$\mathbf{W}^{-1}\frac{\partial\boldsymbol{\Phi}}{\partial\zeta}\mathbf{W} - \frac{\partial\boldsymbol{\Lambda}}{\partial\zeta} = \mathbf{E} \circ (\mathbf{W}^{-1}\frac{\partial\mathbf{W}}{\partial\zeta}). \tag{4.59}$$

After applying the Hadamard product with $\mathbf{I}$ to both sides of (4.59) the forward mode gradient of the eigenvalues is calculated via

$$\frac{\partial\boldsymbol{\Lambda}}{\partial\zeta} = \mathbf{I} \circ (\mathbf{W}^{-1}\frac{\partial\boldsymbol{\Phi}}{\partial\zeta}\mathbf{W}), \tag{4.60}$$

because $\mathbf{E} \circ \mathbf{I} = 0$. Elementwise multiplication of (4.59) with $(\mathbf{F}\circ)$, which is the Hadamard inverse of $\mathbf{E}$ except on the diagonal where it is zero, leads to

$$\underbrace{\mathbf{F} \circ \mathbf{E}}_{\mathbf{1}-\mathbf{I}}\circ(\mathbf{W}^{-1}\frac{\partial\mathbf{W}}{\partial\zeta}) = \mathbf{F} \circ \left(\mathbf{W}^{-1}\frac{\partial\boldsymbol{\Phi}}{\partial\zeta}\mathbf{W}\right) - \underbrace{\mathbf{F} \circ \frac{\partial\boldsymbol{\Lambda}}{\partial\zeta}}_{0}. \tag{4.61}$$

The diagonal elements of $\mathbf{W}^{-1}\frac{\partial\mathbf{W}}{\partial\zeta}$ have to be zero, because a change of ζ can not influence the amplitude of the eigenvectors [Gil08]. Therefore $\mathbf{I} \circ (\mathbf{W}^{-1}\frac{\partial\mathbf{W}}{\partial\zeta})$ is zero and

$$\mathbf{W}^{-1}\frac{\partial\mathbf{W}}{\partial\zeta} = \mathbf{F} \circ \left(\mathbf{W}^{-1}\frac{\partial\boldsymbol{\Phi}}{\partial\zeta}\mathbf{W}\right), \tag{4.62}$$

$$\frac{\partial\mathbf{W}}{\partial\zeta} = \mathbf{W}\left[\mathbf{F} \circ \left(\mathbf{W}^{-1}\frac{\partial\boldsymbol{\Phi}}{\partial\zeta}\mathbf{W}\right)\right] \tag{4.63}$$

denotes the forward mode gradient of the eigenvectors.

To calculate the backward mode gradient of the eigendecomposition, which refers to the derivative of the objective function w.r.t. the PSD matrix, we have to go all the way back through the forward mode calculation. Since (3.10) can be reformulated to

$$\frac{\partial J}{\partial z^*} = \mathrm{Tr}\left(\left(\frac{\partial J}{\partial \mathbf{\Phi}^*}\right)^{\mathrm{H}} \frac{\partial \mathbf{\Phi}}{\partial z^*}\right) + \mathrm{Tr}\left(\left(\frac{\partial J}{\partial \mathbf{\Phi}^*}\right)^{\mathrm{T}} \left(\frac{\partial \mathbf{\Phi}}{\partial z}\right)^*\right)$$
$$= \mathrm{Tr}\left(\left(\frac{\partial J}{\partial \mathbf{\Phi}^*}\right)^{\mathrm{H}} \frac{\partial \mathbf{\Phi}}{\partial z^*}\right) + \mathrm{Tr}\left(\left(\frac{\partial J}{\partial \mathbf{\Phi}^*}\right)^{\mathrm{H}} \frac{\partial \mathbf{\Phi}}{\partial z}\right)^*.$$
(4.64)

it is sufficient to focus on the first trace

$$\frac{\partial J}{\partial z^*} = \mathrm{Tr}\left(\left(\frac{\partial J}{\partial \mathbf{\Phi}^*}\right)^{\mathrm{H}} \frac{\partial \mathbf{\Phi}}{\partial z^*}\right) + \dots$$
(4.65)

because the differentiation w.r.t. z leads to the same solution. The dots represent the neglected differentiation w.r.t. z. The chain rule (3.10) also holds for two intermediate matrices:

$$\frac{\partial J}{\partial z^*} = \mathrm{Tr}\left(\left(\frac{\partial J}{\partial \mathbf{W}^*}\right)^{\mathrm{H}} \frac{\partial \mathbf{W}}{\partial z^*} + \left(\frac{\partial J}{\partial \mathbf{\Lambda}^*}\right)^{\mathrm{H}} \frac{\partial \mathbf{\Lambda}}{\partial z^*}\right) + \dots$$
(4.66)

Now we can compare the coefficients of (4.65) with (4.66) to calculate $\frac{\partial J}{\partial \mathbf{\Phi}^*}$. Inserting the forward mode gradients (4.60) and (4.63)

$$\frac{\partial J}{\partial z^*} = \mathrm{Tr}\left(\left(\frac{\partial J}{\partial \mathbf{W}^*}\right)^{\mathrm{H}} \mathbf{W}\left[\mathbf{F} \circ \left(\mathbf{W}^{-1}\frac{\partial \mathbf{\Phi}}{\partial z^*}\mathbf{W}\right)\right]\right)$$
$$+ \mathrm{Tr}\left(\left(\frac{\partial J}{\partial \mathbf{\Lambda}^*}\right)^{\mathrm{H}} \mathbf{I} \circ (\mathbf{W}^{-1}\frac{\partial \mathbf{\Phi}}{\partial z^*}\mathbf{W})\right) + \dots$$
(4.67)

and rearranging the matrices with

$$\mathrm{Tr}\left(AB\right) = \mathrm{Tr}\left(BA\right),$$
(4.68)
$$\mathrm{Tr}\left(A(B \circ C)\right) = \mathrm{Tr}\left((A \circ B^{\mathrm{T}})C\right) \quad [\text{Mil07, Theorem 2.6}],$$
(4.69)

leads to

$$\frac{\partial J}{\partial z^*} = \mathrm{Tr}\left(\mathbf{W}\left[\left(\frac{\partial J}{\partial \mathbf{W}^*}\right)^{\mathrm{H}} \mathbf{W} \circ \mathbf{F}^{\mathrm{T}} + \left(\frac{\partial J}{\partial \mathbf{\Lambda}^*}\right)^{\mathrm{H}}\right]\mathbf{W}^{-1}\frac{\partial \mathbf{\Phi}}{\partial z^*}\right) + \dots$$
(4.70)

The derivation of the objective function w.r.t. $\mathbf{\Phi}^*$ can now be obtained via comparing the coefficients. This leads to the backward mode gradient of the eigendecomposition

$$\frac{\partial J}{\partial \mathbf{\Phi}^*} = \left(\mathbf{W}\left[\left(\frac{\partial J}{\partial \mathbf{W}^*}\right)^{\mathrm{H}} \mathbf{W} \circ \mathbf{F}^{\mathrm{T}} + \left(\frac{\partial J}{\partial \mathbf{\Lambda}^*}\right)^{\mathrm{H}}\right]\mathbf{W}^{-1}\right)^{\mathrm{H}}$$
$$= \mathbf{W}^{-\mathrm{H}}\left[\frac{\partial J}{\partial \mathbf{\Lambda}^*} + \mathbf{F}^* \circ \mathbf{W}^{\mathrm{H}}\frac{\partial J}{\partial \mathbf{W}^*}\right]\mathbf{W}^{\mathrm{H}},$$
(4.71)

where $(\cdot)^{-\mathrm{H}}$ indicates the inverse of the conjugate transpose. In [Gil08] the real-valued version of this equation can be found.

4.17.1 Extension of the gradient

The magnitude of the eigenvectors is one degree of freedom of the eigendecomposition. It is set to one in many numerical implementations. If the following calculations depend on the magnitude of the eigenvectors and $\boldsymbol{\Phi}$ is not a hermitian matrix this has to be considered in the backward gradient. Here we use the normalization for vectors (section 4.12)

$$\mathbf{a} = \frac{\mathbf{w}}{\sqrt{\mathbf{w}^{\mathrm{H}}\mathbf{w}}}, \tag{4.72}$$

$$\frac{\partial J}{\partial \mathbf{w}^*} = \frac{\frac{\partial J}{\partial \mathbf{a}^*} - \mathbf{w}\frac{\Re\{\mathbf{w}^{\mathrm{H}}\frac{\partial J}{\partial \mathbf{a}^*}\}}{\mathbf{w}^{\mathrm{H}}\mathbf{w}}}{\sqrt{\mathbf{w}^{\mathrm{H}}\mathbf{w}}}. \tag{4.73}$$

If $\mathbf{w}$ has already the magnitude one ($\mathbf{a} = \mathbf{w}$), this simplifies the equation to

$$\frac{\partial J}{\partial \mathbf{w}^*} = \frac{\partial J}{\partial \mathbf{a}^*} - \mathbf{w}\Re\{\mathbf{w}^{\mathrm{H}}\frac{\partial J}{\partial \mathbf{a}^*}\}. \tag{4.74}$$

The extension to all eigenvectors $\mathbf{W} = [\mathbf{w}_1, \ldots, \mathbf{w}_N]$ leads to

$$\frac{\partial J}{\partial \mathbf{W}^*} = \frac{\partial J}{\partial \mathbf{A}^*} - \mathbf{W}\left(\Re\{\mathbf{W}^{\mathrm{H}}\frac{\partial J}{\partial \mathbf{A}^*}\} \circ \mathbf{I}\right) \tag{4.75}$$

and the complete gradient of the eigendecomposition is

$$\begin{aligned}\frac{\partial J}{\partial \boldsymbol{\Phi}^*} &= \mathbf{W}^{-\mathrm{H}}\left(\frac{\partial J}{\partial \boldsymbol{\Lambda}^*} + \mathbf{F}^* \circ \left(\mathbf{W}^{\mathrm{H}}\frac{\partial J}{\partial \mathbf{A}^*}\right)\right)\mathbf{W}^{\mathrm{H}} \\ &\quad - \mathbf{W}^{-\mathrm{H}}\left(\mathbf{F}^* \circ \mathbf{W}^{\mathrm{H}}\mathbf{W}\left(\Re\{\mathbf{W}^{\mathrm{H}}\frac{\partial J}{\partial \mathbf{A}^*}\} \circ \mathbf{I}\right)\right)\mathbf{W}^{\mathrm{H}}.\end{aligned} \tag{4.76}$$

Since (4.73) is not actually calculated during the forward step, we can relabel $\frac{\partial J}{\partial \mathbf{A}^*}$ back into $\frac{\partial J}{\partial \mathbf{W}^*}$

$$\begin{aligned}\frac{\partial J}{\partial \boldsymbol{\Phi}^*} &= \mathbf{W}^{-\mathrm{H}}\left(\frac{\partial J}{\partial \boldsymbol{\Lambda}^*} + \mathbf{F}^* \circ \left(\mathbf{W}^{\mathrm{H}}\frac{\partial J}{\partial \mathbf{W}^*}\right)\right)\mathbf{W}^{\mathrm{H}} \\ &\quad - \mathbf{W}^{-\mathrm{H}}\left(\mathbf{F}^* \circ \mathbf{W}^{\mathrm{H}}\mathbf{W}\left(\Re\{\mathbf{W}^{\mathrm{H}}\frac{\partial J}{\partial \mathbf{W}^*}\} \circ \mathbf{I}\right)\right)\mathbf{W}^{\mathrm{H}}.\end{aligned} \tag{4.77}$$

This extension is not always necessary. When $\boldsymbol{\Phi}$ is a hermitian matrix, then the eigenvector matrix $\mathbf{W}$ is an orthogonal matrix ($\mathbf{W}^{-1} = \mathbf{W}^{\mathrm{H}}$). With this constraint

$$\mathbf{I} = \mathbf{W}^{\mathrm{H}}\mathbf{W} \tag{4.78}$$

holds and since $\mathbf{F}$ is an off diagonal matrix, then

$$\mathbf{0} = \mathbf{F}^* \circ \mathbf{I} \circ (\cdot), \tag{4.79}$$

and the extension is zero.

A second case is, when the following calculations do not depend on the magnitude of the eigenvectors. Then the dot product between an eigenvector and its gradient

$$\Re\{\mathbf{w}_n^{\mathrm{H}} \frac{\partial J}{\partial \mathbf{w}_n^*}\} \tag{4.80}$$

is zero, because the projection of the gradient in the direction of the eigenvector is zero, i.e. the angle between them is 90°. Since this values are the diagonal elements of

$$\Re\{\mathbf{W}^{\mathrm{H}} \frac{\partial J}{\partial \mathbf{W}^*}\} \circ \mathbf{I}, \tag{4.81}$$

the extension is zero.

4.17.2 Verification of the gradient

The eigendecomposition is invertible. Therefore the calculation for

$$\mathbf{\Lambda}, \mathbf{W} = \mathrm{eig}(\boldsymbol{\phi}), \tag{4.82}$$

$$\tilde{\boldsymbol{\phi}} = \mathbf{W}\mathbf{\Lambda}\mathbf{W}^{-1}, \tag{4.83}$$

with an arbitrary matrix $\boldsymbol{\phi}$ and backward gradient $\frac{\partial J}{\partial \tilde{\boldsymbol{\phi}}^*}$ should not affect the gradient. The proof of this can be found in Appendix A.4. One conclusion of this calculation is, that the extension of the gradient for this example is redundant, because it is zero. Also in a numerical verification this identity holds.

For the verification of the necessity of the extension the following numerical proof is done, by calculating the numeric analytical and approximated numeric (Section 3.5.3) gradient of

$$\mathbf{\Lambda}, [\mathbf{w}_1, \ldots, \mathbf{w}_N] = \mathrm{eig}(\boldsymbol{\phi}), \tag{4.84}$$

$$\tilde{\mathbf{w}}_n = \mathbf{w}_n \, \mathrm{e}^{-\mathrm{j}\angle w_{1n}} \quad \text{for } n \in \{1, \ldots, N\}, \tag{4.85}$$

where $\angle$ extracts the angle. The second equation removes the degree of freedom in an absolute phase of the eigenvectors. This is necessary, because the approximated numeric gradient requires a similar output for similar inputs. The simulations have shown, that for this test the extended gradient is necessary to achieve the correct gradient.

The third verification is again a numerical test of the gradient, but this time in the context of the GEV beamformer. The GEV beamformer with a unique output can be written as follows:

$$\mathbf{\Lambda}, [\mathbf{w}_1, \ldots, \mathbf{w}_N] = \mathrm{gev}\left(\sum_t M_t^{(X)}\mathbf{Y}_t\mathbf{Y}_t^{\mathrm{H}}, \sum_t M_t^{(N)}\mathbf{Y}_t\mathbf{Y}_t^{\mathrm{H}}\right), \tag{4.86}$$

$$\mathbf{w} = \mathbf{w}_n \, \mathrm{e}^{-\mathrm{j}\angle w_{1n}}, \tag{4.87}$$

where $n = \underset{n}{\operatorname{argmax}} \{\lambda_n\}$ and λ_n are the diagonal entries of $\mathbf{\Lambda}$ and the function gev solves the generalized eigenvalue problem. An alternative calculation of the beamformer is over spatial whitening (Appendix A.5)

$$\mathbf{\Phi}_{NN}{}^{\frac{1}{2}} = \text{cholesky}\left(\sum_t M_t^{(N)} \mathbf{Y}_t \mathbf{Y}_t^{\mathrm{H}}\right), \tag{4.88}$$

$$\tilde{\mathbf{Y}}_t = \mathbf{\Phi}_{NN}{}^{-\frac{1}{2}} \mathbf{Y}_t, \tag{4.89}$$

$$\mathbf{\Lambda}, [\tilde{\mathbf{w}}_1, \dots, \tilde{\mathbf{w}}_N] = \text{eig}\left(\sum_t M_t^{(X)} \tilde{\mathbf{Y}}_t^{(X)} \tilde{\mathbf{Y}}_t^{\mathrm{H}}\right), \tag{4.90}$$

$$\mathbf{w}_n = \left(\mathbf{\Phi}_{NN}{}^{-\frac{1}{2}}\right)^{\mathrm{H}} \tilde{\mathbf{w}}_n, \tag{4.91}$$

$$\mathbf{w} = \frac{\mathbf{w}_n}{|\mathbf{w}_n|} \, \mathrm{e}^{-\mathrm{j}\angle w_{1n}}, \tag{4.92}$$

where only the gradient for the eigenvalue problem for hermitian matrices is required. As it should be, both beamformers are equal and also the gradients for $M_t^{(X)}$, $M_t^{(N)}$ and $\mathbf{Y}_t$ are equal. Of cause the gradient also survives a numerical verification.

4.18 Overview

In the end of this chapter all gradients are summarized in three tables (Tables 4.1 to 4.3). For this overview the notation of the gradients are changed from the explicit notation $\frac{\partial J}{\partial z^*}$ to the shorter notation ∇_{z^*}. Also the alternative gradients $\nabla_z = \frac{\partial J}{\partial z} = \left(\frac{\partial J}{\partial z^*}\right)^*$ are shown.

Table 4.1: Backward gradients of common scalar operations

	Forward	Backward $\nabla_{\square^*}$	Backward $\nabla_{\square}$
Identity (4.2)	$s = z$	$\nabla_{z^*} = \nabla_{s^*}$	$\nabla_z = \nabla_s$
Conjugate (4.4)	$s = z^*$	$\nabla_{z^*} = \left(\nabla_{s^*}\right)^*$	$\nabla_z = \nabla_s^*$
Negation	$s = -z$	$\nabla_{z^*} = -\nabla_{s^*}$	$\nabla_z = -\nabla_s$
Addition (4.6)	$s = z_1 + z_2$	$\nabla_{z_1^*} = \nabla_{s^*}$ $\nabla_{z_2^*} = \nabla_{s^*}$	$\nabla_{z_1} = \nabla_s$ $\nabla_{z_2} = \nabla_s$
Multiplication (4.9)	$s = z_1 z_2$	$\nabla_{z_1^*} = \nabla_{s^*} z_2^*$ $\nabla_{z_2^*} = \nabla_{s^*} z_1^*$	$\nabla_{z_1} = \nabla_s z_2$ $\nabla_{z_2} = \nabla_s z_1$
Exponentiation (4.11)	$s = z^n$	$\nabla_{z^*} = \nabla_{s^*} n \left(z^*\right)^{n-1}$	$\nabla_z = \nabla_s n \left(z\right)^{n-1}$
Division (4.14)	$s = z_1 / z_2$	$\nabla_{z_1^*} = \nabla_{s^*} \frac{1}{z_2^*}$ $\nabla_{z_2^*} = \nabla_{s^*} \frac{-z_1^*}{\left(z_2^*\right)^2}$	$\nabla_{z_1} = \nabla_s / z_2$ $\nabla_{z_2} = -\nabla_s \frac{z_1}{z_2^2}$
Absolute (4.16)	$s = |z|$	$\nabla_{z^*} = \nabla_{s^*}\, \mathrm{e}^{\mathrm{j}\varphi}$	$\nabla_z = \nabla_s\, \mathrm{e}^{-\mathrm{j}\varphi}$
Phase Factor (4.20)	$s = \mathrm{e}^{\mathrm{j}\varphi}$	$\nabla_{z^*} = \nabla_{s^*} \frac{1}{|z|} - \Re\left\{\nabla_{s^*} \frac{1}{z}\right\} \mathrm{e}^{\mathrm{j}\varphi}$	$\nabla_z = \frac{\nabla_s}{|z|} - \Re\left\{\nabla_s \frac{1}{z^*}\right\} \mathrm{e}^{-\mathrm{j}\varphi}$
Real part (4.22)	$s = \Re\left\{z\right\}$	$\nabla_{z^*} = \nabla_{s^*}$	$\nabla_z = \nabla_s$
Imag. part (4.24)	$s = \Im\left\{z\right\}$	$\nabla_{z^*} = \mathrm{j}\nabla_{s^*}$	$\nabla_z = -\mathrm{j}\nabla_s$

Table 4.2: Backward gradients of discrete Fourier transformations

	Forward	Backward $\nabla_{\square^*}$
		Backward $\nabla_{\square}$
DFT (4.26)	$s_f = \underset{n \to f}{\mathrm{DFT}} \{z_n\}$	$\nabla_{z^*} = N \cdot \underset{f \to n}{\mathrm{IDFT}} \left\{ \nabla_{s_f^*} \right\}$
		$\nabla_z = \underset{f \to n}{\mathrm{DFT}} \left\{ \nabla_{s_f} \right\}$
IDFT (4.28)	$s_f = \underset{f \to n}{\mathrm{IDFT}} \{z_n\}$	$\nabla_{z^*} = \frac{1}{N} \cdot \underset{n \to f}{\mathrm{DFT}} \left\{ \nabla_{s_n^*} \right\}$
		$\nabla_z = \underset{n \to f}{\mathrm{IDFT}} \left\{ \nabla_{s_n} \right\}$
RDFT (4.31) and (4.32)	$s_f = \underset{n \to f}{\mathrm{RDFT}} \{z_n\}$	$\widetilde{\nabla_{s_f^*}} = \begin{cases} \Re\left\{ \nabla_{s_f^*} \right\}, & \text{for } f \in \left\{ 0, \frac{N}{2} \right\} \\ \frac{1}{2} \nabla_{s_f^*}, & \text{for } f \in \left[1, \ldots, \frac{N}{2} - 1 \right] \end{cases}$ $\nabla_{z_n^*} = N \cdot \underset{f \to n}{\mathrm{IDFT}} \left\{ \widetilde{\nabla_{s_f^*}} \right\}$
		$\widetilde{\nabla_{s_f}} = \begin{cases} \Re\left\{ \nabla_{s_f} \right\}, & \text{for } f \in \left\{ 0, \frac{N}{2} \right\} \\ \frac{1}{2} \nabla_{s_f}, & \text{for } f \in \left[1, \ldots, \frac{N}{2} - 1 \right] \end{cases}$ $\nabla_{z_n} = \underset{f \to n}{\mathrm{DFT}} \left\{ \widetilde{\nabla_{s_f}} \right\}$
IRDFT (4.34) and (4.35)	$s_f = \underset{f \to n}{\mathrm{IRDFT}} \{z_n\}$	$\widetilde{\nabla_{z_f^*}} = \frac{1}{N} \cdot \underset{n \to f}{\mathrm{DFT}} \left\{ \nabla_{s_n^*} \right\}$ $\nabla_{z_f^*} = \begin{cases} \widetilde{\nabla_{z_f^*}}, & \text{for } f \in \left\{ 0, \frac{N}{2} \right\} \\ 2\widetilde{\nabla_{z_f^*}}, & \text{for } f \in \left[1, \ldots, \frac{N}{2} - 1 \right] \end{cases}$
		$\widetilde{\nabla_{z_f}} = \underset{n \to f}{\mathrm{IDFT}} \left\{ \nabla_{s_n} \right\}$ $\nabla_{z_f} = \begin{cases} \widetilde{\nabla_{z_f}}, & \text{for } f \in \left\{ 0, \frac{N}{2} \right\} \\ 2\widetilde{\nabla_{z_f}}, & \text{for } f \in \left[1, \ldots, \frac{N}{2} - 1 \right] \end{cases}$

Table 4.3: Backward gradients of common matrix operations

	Forward	Backward $\nabla_{\square^*}$
		Backward $\nabla_{\square}$
Vector Normalization (4.38)	$\mathbf{s} = \frac{\mathbf{z}}{\sqrt{\mathbf{z}^H \mathbf{z}}}$	$\nabla_{\mathbf{z}^*} = \frac{\nabla_{\mathbf{s}^*} - \frac{\mathbf{z}}{\mathbf{z}^H \mathbf{z}} \Re\{\mathbf{z}^H \nabla_{\mathbf{s}^*}\}}{\sqrt{\mathbf{z}^H \mathbf{z}}}$
		$\nabla_{\mathbf{z}} = \frac{\nabla_{\mathbf{s}} - \frac{\mathbf{z}^*}{\mathbf{z}^H \mathbf{z}} \Re\{\mathbf{z}^T \nabla_{\mathbf{s}}\}}{\sqrt{\mathbf{z}^H \mathbf{z}}}$
Matrix Multiplication (4.42) and (4.43)	$\mathbf{S} = \mathbf{Z}_1 \mathbf{Z}_2$	$\nabla_{\mathbf{z}_1^*} = \nabla_{\mathbf{S}^*} \mathbf{Z}_2^H$ $\nabla_{\mathbf{z}_2^*} = \mathbf{Z}_1^H \nabla_{\mathbf{S}^*}$
		$\nabla_{\mathbf{z}_1} = \nabla_{\mathbf{S}} \mathbf{Z}_2^T$ $\nabla_{\mathbf{z}_2} = \mathbf{Z}_1^T \nabla_{\mathbf{S}}$
Matrix Inversion (4.45)	$\mathbf{S} = \mathbf{Z}^{-1}$	$\nabla_{\mathbf{z}^*} = -\mathbf{S}^H \nabla_{\mathbf{S}^*} \mathbf{S}^H$
		$\nabla_{\mathbf{z}} = -\mathbf{S}^T \nabla_{\mathbf{S}} \mathbf{S}^T$
Matrix Inverse product 1 (4.47) and (4.48)	$\mathbf{S} = \mathbf{Z}_1^{-1} \mathbf{Z}_2$	$\nabla_{\mathbf{z}_2^*} = \mathbf{Z}_1^{-H} \nabla_{\mathbf{S}^*}$ $\nabla_{\mathbf{z}_1^*} = -\nabla_{\mathbf{z}_2^*} \mathbf{S}^H$
		$\nabla_{\mathbf{z}_2} = \mathbf{Z}_1^{-T} \nabla_{\mathbf{S}}$ $\nabla_{\mathbf{z}_1} = -\nabla_{\mathbf{z}_2} \mathbf{S}^T$
Matrix Inverse product 2 (4.50) and (4.51)	$\mathbf{S} = \mathbf{Z}_1 \mathbf{Z}_2^{-1}$	$\nabla_{\mathbf{z}_1^*} = \nabla_{\mathbf{S}^*} \mathbf{Z}_2^{-H}$ $\nabla_{\mathbf{z}_2^*} = -\mathbf{S}^H \nabla_{\mathbf{z}_1^*}$
		$\nabla_{\mathbf{z}_1} = \nabla_{\mathbf{S}} \mathbf{Z}_2^{-T}$ $\nabla_{\mathbf{z}_2} = -\mathbf{S}^T \nabla_{\mathbf{z}_1}$
Cholesky decomposition (4.53) and (4.54)	$\mathbf{S} = \mathbf{g}(\mathbf{Z})$	$\nabla_{\tilde{\mathbf{z}}^*} = \mathbf{S}^{-H} \left((\mathbf{S}^H \nabla_{\mathbf{S}^*}) \circ (\triangle - \frac{1}{2}\mathbf{I}) \right) \mathbf{S}^{-1}$ $\nabla_{\mathbf{z}^*} = \frac{1}{2} \left(\nabla_{\tilde{\mathbf{z}}^*} + (\nabla_{\tilde{\mathbf{z}}^*})^H \right)$
		$\nabla_{\tilde{\mathbf{z}}} = \mathbf{S}^{-T} \left((\mathbf{S}^T \nabla_{\mathbf{S}}) \circ (\triangle - \frac{1}{2}\mathbf{I}) \right) (\mathbf{S}^{-1})^*$ $\nabla_{\mathbf{z}} = \frac{1}{2} \left(\nabla_{\tilde{\mathbf{z}}} + (\nabla_{\tilde{\mathbf{z}}})^H \right)$
Eigenvalue decomposition (4.77)	$\mathbf{\Lambda}, \mathbf{S} = \mathrm{eig}(\mathbf{Z})$	$\nabla_{\mathbf{z}^*} = \mathbf{S}^{-H} \left(\nabla_{\mathbf{\Lambda}^*} + \mathbf{F}^* \circ (\mathbf{S}^H \nabla_{\mathbf{S}^*}) \right) \mathbf{S}^H$ $\quad - \mathbf{S}^{-H} \left(\mathbf{F}^* \circ \mathbf{S}^H \mathbf{S} \left(\Re\{\mathbf{S}^H \nabla_{\mathbf{S}^*}\} \circ \mathbf{I} \right) \right) \mathbf{S}^H$ $F_{ij} = (\lambda_j - \lambda_i)^{-1} \quad \forall_{i \neq j} \quad \text{and} \quad F_{ii} = 0$
		$\nabla_{\mathbf{z}} = \mathbf{S}^{-T} \left(\nabla_{\mathbf{\Lambda}} + \mathbf{F} \circ (\mathbf{S}^T \nabla_{\mathbf{S}}) \right) \mathbf{S}^T$ $\quad - \mathbf{S}^{-T} \left(\mathbf{F} \circ \mathbf{S}^T \mathbf{S}^* \left(\Re\{\mathbf{S}^T \nabla_{\mathbf{S}}\} \circ \mathbf{I} \right) \right) \mathbf{S}^T$

5 Evaluation

To evaluate the approach described in Chapter 2 two different performance measures are used. The quality of the beamformed signal is evaluated with the perceptual evaluation of speech quality [Rix+01] (PESQ) measure. Also the SNR is calculated. Both measures are described in the following sections. Afterwards the system environment and hyperparameters as well as the simulation procedure are denoted. In the end an outlook to an end-to-end system evaluated with the word error rate (WER) is given.

5.1 Perceptual Evaluation of Speech Quality (PESQ)

In the ITU-T-Recommendation P.862 (International Telecommunication Union) and [Rix+01] the PESQ procedure for wideband signals is described. It compares a transmitted with a received signal via an psychoacoustic model. The measurement is done in MOS-LQO (Mean Opinion Score - Listening Quality Objective). The MOS-Scale is defined as follows: 5 = Excellent, 4 = Good, 3 = Fair, 2 = Poor, 1 = Bad.
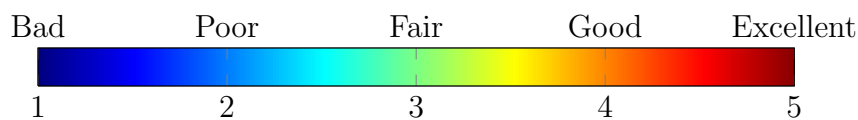

Figure 5.1: MOS-Scale

5.2 Signal-to-noise ratio (SNR)

The SNR Criterion describes the ratio between the useful signal power and the noise power. The input SNR

$$\text{SNR}_{\text{in}} = 10 \log_{10} \frac{\sum_{t=1}^{T} \sum_{f=1}^{F} ||\mathbf{X}_{f,t}||^2}{\sum_{t=1}^{T} \sum_{f=1}^{F} ||\mathbf{N}_{f,t}||^2} \tag{5.1}$$

is the ratio between the clean signal and the noise and the output SNR in a beamformer scenario

$$\mathrm{SNR_{out}} = 10 \log_{10} \frac{\sum_{t=1}^{T} \sum_{f=1}^{F} \left| \mathbf{w}_f^{\mathrm{H}} \mathbf{X}_{f,t} \right|^2}{\sum_{t=1}^{T} \sum_{f=1}^{F} \left| \mathbf{w}_f^{\mathrm{H}} \mathbf{N}_{f,t} \right|^2} \qquad (5.2)$$

the ratio of the beamformed clean signals and the beamformed noise.

5.3 Environment and hyperparameter

- Database: CHiME-3 [Bar+15]
    - Sets:
        * Train: *tr05_simu* with 7138 simulated utterances
        * cross validation: *dt05_simu* with 1640 simulated utterances
        * evaluation: *dt05_simu* with 1640 simulated utterances
    - Microphones:
        * Count: 6
        * Positions: fixed
        * Speaker positions: low variation
    - Locations:
        * bus
        * cafe
        * pedestrian area
        * street junction
    - Text prompts:
        * Language: English
        * Origin: Wall Street Journal (WSJ0) corpus
- Sampling frequency: 16 000 Hz
- STFT parameters
    - FFT length: 1024 (= 64 ms
    - FFT shift: 256 (= 16 ms, $\frac{1}{4}$ of the length)
    - window: blackman

Table 5.1: Configuration of the neural network

	Units	Type	Non-Linearity	p_{dropout}
L1	256	BLSTM	Tanh	0.5
L2	513	FF	ReLU	0.5
L3	513	FF	ReLU	0.5
L4	1026	FF	Sigmoid	0.0

For the evaluation of the NN we used the same Dataset as for cross validation. The official evaluation dataset of CHiME-3 can not be used, because it does not contain the clean signal, which is required for the performance measurements.

5.4 Simulation

To evaluate the complex gradients, they are tested against [HDHU16], which uses the NN for speech enhancement (see Table 5.1). In [HDHU16] the objective function is the binary cross entropy of the network output with a heuristic binary mask. The objective function in this report is replaced with an SNR objective, like described in (2.5).

Table 5.2 summarizes the simulation results. The first row shows the input PESQ and SNR. The first column indicates the origin of the masks, where Oracle [HDHU16] is the heuristic binary mask, [HDHU16] the network trained with the binary cross entropy objective and NN-GEV and NN-MVDR use mask estimation networks trained with an SNR objective, where the first uses a GEV beamformer and the second an MVDR beamformer. Note that the beamformer used during NN training (denoted "mask sourse" in Table 5.2) can be different from the evaluation beamformer. The optional post filter blind analytic normalization (BAN) is described in [WHU07] and $|| \cdot ||_2$ is a normalization to unit length.

The proposed alternative objective with a GEV beamformer outperforms the reference [HDHU16] both in PESQ and SNR. Unexpectedly the GEV beamformer (also known as max SNR beamformer) nearly reaches the oracle perfomance in PESQ, but not the oracle SNR. The MVDR beamformer as objective is not able to beat the reference with the MVDR evaluation beamformer. With an GEV evaluation the results get better, but they do not exceed the NN-GEV.

In Fig. 5.2 some masks at the output of the NN are shown exemplary, where Figs. 5.2a and 5.2b are from [HDHU16] and NN-GEV, respectively. The evaluation beamformer for the scores is a GEV with a BAN postfilter in both cases. Counter-intuitively the right mask yields slightly better scores, although no expected symmetries as in the left one are visible. Also the mask tends to suddenly change its values between the frequencies.

(a) [HDHU16] (PESQ: 2,0; SNR: 23,6)   (b) NN-GEV (PESQ: 2,3; SNR: 27,7)
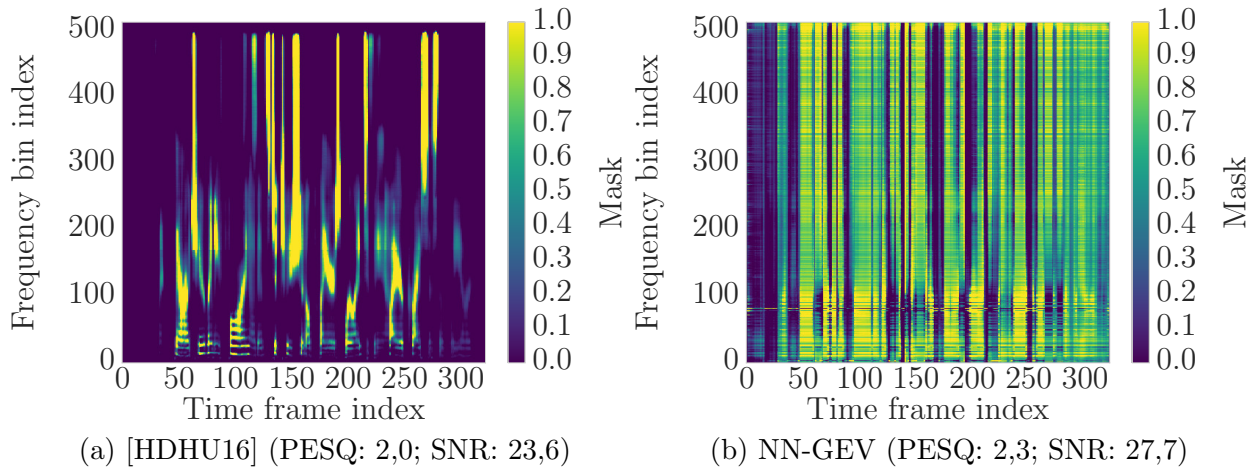
Figure 5.2: Sample output masks for reference and NN-GEV[1]

Table 5.2: Overview of the results for different system configurations. The first row shows the scores of the input noisy signal.

mask source	evaluation beamformer	post filter	PESQ	SNR
-	-	-	1,21	3,85
Oracle [HDHU16]	GEV	-	1,73	15,35
		BAN	1,76	18,02
[HDHU16]	GEV	-	1,64	14,08
		BAN	1,70	16,03
NN-GEV	GEV	-	1,71	14,59
		BAN	1,75	16,05
NN-MVDR	MVDR	-	1,43	10,58
		$\|\cdot\|_2$	1,60	13,84
	GEV	-	1,60	14,21
		BAN	1,50	14,51

---

[1]The CHiME-3 utterance ID is `f04_051c0112_str`.

5.5 Outlook: ASR end-to-end

Based on this work in [Hey+17] further simulations with a backpropagation through beamformers are done. Here only a small summary is presented. There the larger system (see Fig. 2.1 purple part) is described with two NNs in an ASR system. The two NNs are one beamforming NN (BF), which is the same as described here, and an acoustic model NN (AM). Table 5.3 shows the WER from [Hey+17].

The training is done with different configurations:

- fixed: Each network is trained separately, without knowing the other NN. This means that the AM is trained on each channel of the noisy signal.

- scratch: Networks are randomly initialized and only trained in an end-to-end configuration, where the gradient of the AM is passed to the mask estimation NN.

- finetune: The networks are first trained separately an then finetuned in an end-to-end configuration.

Table 5.3: Average WER (%) for the described systems in [Hey+17]. Copied with permission from [Hey+17].

Training		Dev		Test	
BF	AM	real	simu	real	simu
fixed	fixed	4.26	4.29	5.85	4.59
scratch	scratch	5.51	5.19	8.76	5.61
scratch	finetune	4.14	4.09	5.86	4.06
fixed	finetune	4.09	3.96	5.56	3.9
finetune	finetune	3.77	3.89	5.42	3.95

While pure end-to-end training was not able to increase the performance, finetuning the backprobagation through the beamformer outperforms the reference system.

6 Summary

In this report the derivation of complex-valued gradients used in NNs is described. Many well-known real-valued gradients are extended to their complex-valued counterparts. This enables the use of complex-valued operations in the calculation of the objective function of a NN. Here the optimization of a mask estimation NN w.r.t. the beamformer output SNR is shown exemplarily. The changed objective function leads to a gain in PESQ and SNR in comparison to a objective function based on a heuristic reference mask. Also two different statistically optimal beamformer operations are compared.

As a key theoretical result the derivation of the gradient of the eigenvalue problem with complex-valued eigenvalues is derived, which is an extension of the known gradient for hermitian matrices. Also the complex-valued gradients of the Cholesky decomposition, matrix product and inverse, DFT and vector normalization are derived. So this report can be treated as a collection of complex-valued gradients, which can be used in the calculation of a NN objective function, too.

Using these results the complete ASR system with a mask estimation NN and a multi-channel beamformer as a speech enhancement preprocessing can be optimized together w.r.t. to a common objective. The results of simulations described in Section 5.5 show that pretrained networks fine-tuned with the backpropagation trough the beamformer leads to better WERs.

A Appendix

A.1 Proof of the matrix product rule

$$\frac{\partial AB}{\partial \zeta} = \frac{\partial A}{\partial \zeta}B + A\frac{\partial B}{\partial \zeta} \qquad \zeta \in z, z^* \tag{A.1}$$

Proof. With $C, D, G, H \in \mathbb{R}$ and

$$A = C + \mathrm{j}D, \quad B = G + \mathrm{j}H, \quad AB = CG - DH + \mathrm{j}\left(DG + CH\right) \tag{A.2}$$

$$
\begin{aligned}
\frac{\partial AB}{\partial \zeta} =& \frac{1}{2}\left(\frac{\partial AB}{\partial x} \pm \mathrm{j}\frac{\partial AB}{\partial y}\right) =
\begin{cases}
\frac{1}{2}\left(\frac{\partial AB}{\partial x} + \mathrm{j}\frac{\partial AB}{\partial y}\right) & \text{for } \zeta = z^* \\
\frac{1}{2}\left(\frac{\partial AB}{\partial x} - \mathrm{j}\frac{\partial AB}{\partial y}\right) & \text{for } \zeta = z
\end{cases} \\
=& \frac{1}{2}\left(\frac{\partial}{\partial x}\left(CG - DH\right) \mp \frac{\partial}{\partial y}\left(DG + CH\right) + \mathrm{j}\left(\frac{\partial}{\partial x}\left(DG + CH\right) \pm \frac{\partial}{\partial y}\left(CG - DH\right)\right)\right) \\
=& \frac{1}{2}\left(\frac{\partial C}{\partial x}G + C\frac{\partial G}{\partial x} - \frac{\partial D}{\partial x}H - D\frac{\partial H}{\partial x} \mp \frac{\partial D}{\partial y}G \mp D\frac{\partial G}{\partial y} \mp \frac{\partial C}{\partial y}H \mp c\frac{\partial H}{\partial y}\right) \\
&\pm \frac{1}{2}\mathrm{j}\left(\frac{\partial D}{\partial x}G + D\frac{\partial G}{\partial x} + \frac{\partial C}{\partial x}H + C\frac{\partial H}{\partial x} \pm \frac{\partial C}{\partial y}G \pm C\frac{\partial G}{\partial y} \mp \frac{\partial D}{\partial y}H \mp D\frac{\partial H}{\partial y}\right) \\
=& \frac{1}{2}\left(\frac{\partial C}{\partial x}\underbrace{\left(G + \mathrm{j}H\right)}_{B} - \frac{\partial D}{\partial x}\underbrace{\left(H - \mathrm{j}G\right)}_{-\mathrm{j}B} \mp \frac{\partial D}{\partial y}\underbrace{\left(G + \mathrm{j}H\right)}_{B} \mp \frac{\partial C}{\partial y}\underbrace{\left(H - \mathrm{j}G\right)}_{-\mathrm{j}B}\right) \\
&+ \frac{1}{2}\left(\underbrace{\left(C + \mathrm{j}D\right)}_{A}\frac{\partial G}{\partial x} - \underbrace{\left(D - \mathrm{j}C\right)}_{-\mathrm{j}A}\frac{\partial H}{\partial x} \mp \underbrace{\left(D - \mathrm{j}C\right)}_{-\mathrm{j}A}\frac{\partial G}{\partial y} \mp \underbrace{\left(C + \mathrm{j}D\right)}_{A}\frac{\partial H}{\partial y}\right) \\
=& \frac{1}{2}\left(\frac{\partial C}{\partial x} + \mathrm{j}\frac{\partial D}{\partial x} \mp \frac{\partial D}{\partial y} \pm \mathrm{j}\frac{\partial C}{\partial y}\right)B + \frac{1}{2}A\left(\frac{\partial G}{\partial x} + \mathrm{j}\frac{\partial H}{\partial x} \mp \frac{\partial H}{\partial y} \pm \mathrm{j}\frac{\partial G}{\partial y}\right) \\
=& \frac{\partial A}{\partial \zeta}B + A\frac{\partial B}{\partial \zeta}
\end{aligned}
\tag{A.3}
$$

∎

A.2 Proof of complex-valued chain rule

Proof. For the verification of (3.7)

$$
\frac{\partial J}{\partial z^*} = \frac{1}{2}\left(\frac{\partial J}{\partial \sigma}\frac{\partial \sigma}{\partial x} + \frac{\partial J}{\partial \omega}\frac{\partial \omega}{\partial x} + j\frac{\partial J}{\partial \sigma}\frac{\partial \sigma}{\partial y} + j\frac{\partial J}{\partial \omega}\frac{\partial \omega}{\partial y}\right)
$$

$$
\overset{!}{=} \left(\frac{\partial J}{\partial s^*}\right)^*\frac{\partial s}{\partial z^*} + \frac{\partial J}{\partial s^*}\left(\frac{\partial s}{\partial z}\right)^*,
$$

it is easier to start with the right side of the equation, inserting (3.2) and simplify it. This leads to:

$$
4\left(\frac{\partial J}{\partial s^*}\right)^*\frac{\partial s}{\partial z^*} + 4\frac{\partial J}{\partial s^*}\left(\frac{\partial s}{\partial z^*}\right)
$$

$$
= \frac{4}{4}\left(\frac{\partial J}{\partial \sigma} - j\frac{\partial J}{\partial \omega}\right)\left(\frac{\partial s}{\partial x} + j\frac{\partial s}{\partial y}\right) + \frac{4}{4}\left(\frac{\partial J}{\partial \sigma} + j\frac{\partial J}{\partial \omega}\right)\left(\frac{\partial s^*}{\partial x} + j\frac{\partial s^*}{\partial y}\right),
$$

$$
= \frac{\partial J}{\partial \sigma}\frac{\partial s}{\partial x} + \frac{\partial J}{\partial \omega}\frac{\partial s}{\partial y} + j\frac{\partial J}{\partial \sigma}\frac{\partial s}{\partial y} - j\frac{\partial J}{\partial \omega}\frac{\partial s}{\partial x} + \frac{\partial J}{\partial \sigma}\frac{\partial s^*}{\partial x} - \frac{\partial J}{\partial \omega}\frac{\partial s^*}{\partial y} + j\frac{\partial J}{\partial \sigma}\frac{\partial s^*}{\partial y} + j\frac{\partial J}{\partial \omega}\frac{\partial s^*}{\partial x},
$$

$$
= \frac{\partial J}{\partial \sigma}\left(\frac{\partial s}{\partial x} + \frac{\partial s^*}{\partial x}\right) + \frac{\partial J}{\partial \omega}\left(\frac{\partial s}{\partial y} - \frac{\partial s^*}{\partial y}\right) + j\frac{\partial J}{\partial \sigma}\left(\frac{\partial s}{\partial y} + \frac{\partial s^*}{\partial y}\right) - j\frac{\partial J}{\partial \omega}\left(\frac{\partial s}{\partial x} - \frac{\partial s^*}{\partial x}\right),
$$

$$
= \frac{\partial J}{\partial \sigma}\frac{\partial}{\partial x}\underbrace{(s + s^*)}_{2\sigma} + \frac{\partial J}{\partial \omega}\frac{\partial}{\partial y}\underbrace{(s - s^*)}_{j2\omega} + j\frac{\partial J}{\partial \sigma}\frac{\partial}{\partial y}\underbrace{(s + s^*)}_{2\sigma} - j\frac{\partial J}{\partial \omega}\frac{\partial}{\partial x}\underbrace{(s - s^*)}_{j2\omega},
$$

$$
= 2\left(\frac{\partial J}{\partial \sigma}\frac{\partial \sigma}{\partial x} + \frac{\partial J}{\partial \omega}\frac{\partial \omega}{\partial x} + j\frac{\partial J}{\partial \sigma}\frac{\partial \sigma}{\partial y} + j\frac{\partial J}{\partial \omega}\frac{\partial \omega}{\partial y}\right).
$$

$$
(A.4)
$$

■

A.3 Verification of a matrix identity for the calculation of the eigendecomposition gradient

We want to prove:

$$\mathbf{C}\mathbf{\Lambda} - \mathbf{\Lambda}\mathbf{C} = \mathbf{C} \circ \mathbf{E}, \tag{A.5}$$

$$\tag{A.6}$$

where

$$\mathbf{\Lambda} = \begin{bmatrix} \lambda_1 & 0 & \dots \\ 0 & \lambda_2 & \\ \vdots & & \ddots \end{bmatrix} \tag{A.7}$$

$$\mathbf{C} = \begin{bmatrix} c_{11} & c_{21} & \dots \\ c_{21} & c_{22} & \\ \vdots & & \ddots \end{bmatrix} \tag{A.8}$$

$$\mathbf{E} = \begin{bmatrix} \lambda_1 - \lambda_1 & \lambda_2 - \lambda_1 & \dots \\ \lambda_1 - \lambda_2 & \lambda_1 - \lambda_1 & \\ \vdots & & \ddots \end{bmatrix} = \begin{bmatrix} E_{11} & E_{12} & \dots \\ E_{21} & E_{22} & \\ \vdots & & \ddots \end{bmatrix} \tag{A.9}$$

$$E_{ij} = \lambda_j - \lambda_i \tag{A.10}$$

Proof.

$$
\begin{aligned}
\mathbf{C}\mathbf{\Lambda} - \mathbf{\Lambda}\mathbf{C} &= \begin{bmatrix} c_{11} & c_{21} & \dots \\ c_{21} & c_{22} & \\ \vdots & & \ddots \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \dots \\ 0 & \lambda_2 & \\ \vdots & & \ddots \end{bmatrix} - \begin{bmatrix} \lambda_1 & 0 & \dots \\ 0 & \lambda_2 & \\ \vdots & & \ddots \end{bmatrix} \begin{bmatrix} c_{11} & c_{21} & \dots \\ c_{21} & c_{22} & \\ \vdots & & \ddots \end{bmatrix} \\
&= \begin{bmatrix} \lambda_1 c_{11} & \lambda_2 c_{21} & \dots \\ \lambda_1 c_{21} & \lambda_2 c_{22} & \\ \vdots & & \ddots \end{bmatrix} - \begin{bmatrix} \lambda_1 c_{11} & \lambda_1 c_{21} & \dots \\ \lambda_2 c_{21} & \lambda_2 c_{22} & \\ \vdots & & \ddots \end{bmatrix} \\
&= \mathbf{C} \circ \begin{bmatrix} \lambda_1 - \lambda_1 & \lambda_2 - \lambda_1 & \dots \\ \lambda_1 - \lambda_2 & \lambda_2 - \lambda_2 & \\ \vdots & & \ddots \end{bmatrix} \\
&= \mathbf{C} \circ \mathbf{E}
\end{aligned}
$$

∎

A.4 Gradient identity proof for the eigenvalue decomposition

The operation

$$\mathbf{\Lambda}, \mathbf{W} = \text{eig}(\boldsymbol{\phi}), \tag{A.11}$$

$$\tilde{\boldsymbol{\phi}} = \mathbf{W}\mathbf{\Lambda}\mathbf{W}^{-1}, \tag{A.12}$$

does not effect the gradient

$$\frac{\partial J}{\partial \boldsymbol{\phi}^*} \overset{!}{=} \frac{\partial J}{\partial \tilde{\boldsymbol{\phi}}^*}. \tag{A.13}$$

Proof. We start by introducing some auxiliary variables

$$\tilde{\boldsymbol{\phi}} = \mathbf{W}_1 \mathbf{\Lambda} \mathbf{W}_2^{-1} = \mathbf{W}_1 \mathbf{M}. \tag{A.14}$$

$$\tag{A.15}$$

Using the gradients for matrix multiplication (4.42) and (4.43))

$$\frac{\partial J}{\partial \mathbf{W}_1^*} = \frac{\partial J}{\partial \tilde{\boldsymbol{\phi}}^*}\mathbf{M}^{\mathrm{H}} = \frac{\partial J}{\partial \tilde{\boldsymbol{\phi}}^*}\mathbf{W}^{-\mathrm{H}}\mathbf{\Lambda}^{\mathrm{H}}, \tag{A.16}$$

$$\frac{\partial J}{\partial \mathbf{M}^*} = \mathbf{W}_1^{\mathrm{H}}\frac{\partial J}{\partial \tilde{\boldsymbol{\phi}}^*}, \tag{A.17}$$

$$\tag{A.18}$$

the gradient of the matrix inverse multiplication (4.50) and (4.51) under the constraint that $\frac{\partial J}{\partial \mathbf{\Lambda}^*}$ is a diagonal matrix

$$\frac{\partial J}{\partial \mathbf{\Lambda}^*} = \left(\frac{\partial J}{\partial \mathbf{M}^*}\mathbf{W}_2^{-\mathrm{H}}\right) \circ \mathbf{I} = \left(\mathbf{W}^{\mathrm{H}}\frac{\partial J}{\partial \tilde{\boldsymbol{\phi}}^*}\mathbf{W}^{-\mathrm{H}}\right) \circ \mathbf{I}, \tag{A.19}$$

$$\frac{\partial J}{\partial \mathbf{W}_2^*} = -\mathbf{M}^{\mathrm{H}}\frac{\partial J}{\partial \mathbf{M}^*}\mathbf{W}_2^{-\mathrm{H}} = -\mathbf{W}^{-\mathrm{H}}\mathbf{\Lambda}^{\mathrm{H}}\mathbf{W}^{\mathrm{H}}\frac{\partial J}{\partial \tilde{\boldsymbol{\phi}}^*}\mathbf{W}^{-\mathrm{H}}, \tag{A.20}$$

the rule for reusing a variable (Section 3.2)

$$\frac{\partial J}{\partial \mathbf{W}^*} = \frac{\partial J}{\partial \mathbf{W}_1^*} + \frac{\partial J}{\partial \mathbf{W}_2^*} = \frac{\partial J}{\partial \tilde{\boldsymbol{\phi}}^*}\mathbf{W}^{-\mathrm{H}}\mathbf{\Lambda}^{\mathrm{H}} - \mathbf{W}^{-\mathrm{H}}\mathbf{\Lambda}^{\mathrm{H}}\mathbf{W}^{\mathrm{H}}\frac{\partial J}{\partial \tilde{\boldsymbol{\phi}}^*}\mathbf{W}^{-\mathrm{H}}, \tag{A.21}$$

the gradient for the eigenvalue decomposition (4.71)

$$\frac{\partial J}{\partial \tilde{\boldsymbol{\phi}}^*} = \mathbf{W}^{-\mathrm{H}}\left(\frac{\partial J}{\partial \mathbf{\Lambda}^*} + \mathbf{F}^* \circ \left(\mathbf{W}^{\mathrm{H}}\frac{\partial J}{\partial \mathbf{W}^*}\right)\right)\mathbf{W}^{\mathrm{H}} \tag{A.22}$$

$$= \mathbf{W}^{-\mathrm{H}}\left(\frac{\partial J}{\partial \mathbf{\Lambda}^*} + \mathbf{F}^* \circ \left(\mathbf{W}^{\mathrm{H}}\frac{\partial J}{\partial \tilde{\boldsymbol{\phi}}^*}\mathbf{W}^{-\mathrm{H}}\mathbf{\Lambda}^{\mathrm{H}} - \mathbf{W}^{\mathrm{H}}\mathbf{W}^{-\mathrm{H}}\mathbf{\Lambda}^{\mathrm{H}}\mathbf{W}^{\mathrm{H}}\frac{\partial J}{\partial \tilde{\boldsymbol{\phi}}^*}\mathbf{W}^{-\mathrm{H}}\right)\right)\mathbf{W}^{\mathrm{H}} \tag{A.23}$$

$$= \mathbf{W}^{-\mathrm{H}}\left(\frac{\partial J}{\partial \mathbf{\Lambda}^*} + \mathbf{F}^* \circ \left(\mathbf{W}^{\mathrm{H}}\frac{\partial J}{\partial \tilde{\boldsymbol{\phi}}^*}\mathbf{W}^{-\mathrm{H}}\mathbf{\Lambda}^{\mathrm{H}} - \mathbf{\Lambda}^{\mathrm{H}}\mathbf{W}^{\mathrm{H}}\frac{\partial J}{\partial \tilde{\boldsymbol{\phi}}^*}\mathbf{W}^{-\mathrm{H}}\right)\right)\mathbf{W}^{\mathrm{H}} \tag{A.24}$$

and considering that $\mathbf{C}\boldsymbol{\Lambda} - \boldsymbol{\Lambda}\mathbf{C} = \mathbf{E} \circ \mathbf{C}$ (4.57) and $\mathbf{E} \circ \mathbf{F} = \mathbf{1} - \mathbf{I}$ (4.61)

$$\mathbf{C}\boldsymbol{\Lambda}^{\mathrm{H}} - \boldsymbol{\Lambda}^{\mathrm{H}}\mathbf{C} = \mathbf{C}\boldsymbol{\Lambda}^* - \boldsymbol{\Lambda}^*\mathbf{C} = \mathbf{E}^* \circ \mathbf{C} \tag{A.25}$$

$$\frac{\partial J}{\partial \tilde{\boldsymbol{\phi}}^*} = \mathbf{W}^{-\mathrm{H}} \left(\frac{\partial J}{\partial \boldsymbol{\Lambda}^*} + \mathbf{F}^* \circ \mathbf{E}^* \circ \left(\mathbf{W}^{\mathrm{H}} \frac{\partial J}{\partial \tilde{\boldsymbol{\phi}}^*} \mathbf{W}^{-\mathrm{H}} \right) \right) \mathbf{W}^{\mathrm{H}} \tag{A.26}$$

$$= \mathbf{W}^{-\mathrm{H}} \left(\left(\mathbf{W}^{\mathrm{H}} \frac{\partial J}{\partial \tilde{\boldsymbol{\phi}}^*} \mathbf{W}^{-\mathrm{H}} \right) \circ \mathbf{I} + (\mathbf{1} - \mathbf{I}) \circ \left(\mathbf{W}^{\mathrm{H}} \frac{\partial J}{\partial \tilde{\boldsymbol{\phi}}^*} \mathbf{W}^{-\mathrm{H}} \right) \right) \mathbf{W}^{\mathrm{H}}$$
$$\tag{A.27}$$

$$= \mathbf{W}^{-\mathrm{H}} \left(\left(\mathbf{W}^{\mathrm{H}} \frac{\partial J}{\partial \tilde{\boldsymbol{\phi}}^*} \mathbf{W}^{-\mathrm{H}} \right) \circ (\mathbf{I} + \mathbf{1} - \mathbf{I}) \right) \mathbf{W}^{\mathrm{H}} \tag{A.28}$$

$$= \frac{\partial J}{\partial \tilde{\boldsymbol{\phi}}^*} \tag{A.29}$$

verifies that the gradient is not effected by this operation, under the assumption that the gradient extension of eigenvalue decomposition is zero. The assumption holds since

$$\Re \left\{ \mathbf{W}^{\mathrm{H}} \frac{\partial J}{\partial \mathbf{W}^*} \right\} = \Re \left\{ \mathbf{W}^{\mathrm{H}} \frac{\partial J}{\partial \tilde{\boldsymbol{\phi}}^*} \mathbf{W}^{-\mathrm{H}} \boldsymbol{\Lambda}^{\mathrm{H}} - \mathbf{W}^{\mathrm{H}} \mathbf{W}^{-\mathrm{H}} \boldsymbol{\Lambda}^{\mathrm{H}} \mathbf{W}^{\mathrm{H}} \frac{\partial J}{\partial \tilde{\boldsymbol{\phi}}^*} \mathbf{W}^{-\mathrm{H}} \right\}, \tag{A.30}$$

with $\mathbf{I} = \mathbf{W}^{\mathrm{H}}\mathbf{W}^{-\mathrm{H}}$ and $\mathbf{C}\boldsymbol{\Lambda}^{\mathrm{H}} - \boldsymbol{\Lambda}^{\mathrm{H}}\mathbf{C} = \mathbf{E}^* \circ \mathbf{C}$

$$\Re \left\{ \mathbf{W}^{\mathrm{H}} \frac{\partial J}{\partial \mathbf{W}^*} \right\} = \Re \left\{ \mathbf{W}^{\mathrm{H}} \frac{\partial J}{\partial \tilde{\boldsymbol{\phi}}^*} \mathbf{W}^{-\mathrm{H}} \boldsymbol{\Lambda}^{\mathrm{H}} - \boldsymbol{\Lambda}^{\mathrm{H}} \mathbf{W}^{\mathrm{H}} \frac{\partial J}{\partial \tilde{\boldsymbol{\phi}}^*} \mathbf{W}^{-\mathrm{H}} \right\} \tag{A.31}$$

$$= \Re \left\{ \left(\mathbf{W}^{\mathrm{H}} \frac{\partial J}{\partial \tilde{\boldsymbol{\phi}}^*} \mathbf{W}^{-\mathrm{H}} \right) \circ \mathbf{E}^* \right\} \tag{A.32}$$

is a off diagonal matrix (the elements on the diagonal are zero) and appears in the extended gradient $\Re \left\{ \mathbf{W}^{\mathrm{H}} \frac{\partial J}{\partial \mathbf{W}^*} \right\} \circ \mathbf{I}$. Therefore the extended gradient is for this example not necessary.   ∎

A.5 GEV beamformer via spatial whitening

Noise PSD Matrix:

$$\boldsymbol{\Phi}_{NNf} = \frac{\sum_{t=1}^{T} \sum_{d=1}^{D} M_{f,t,d}^{(\nu)} \mathbf{Y}_{f,t} \cdot \mathbf{Y}_{f,t}^{\mathrm{H}}}{\sum_{t=1}^{T} \sum_{d=1}^{D} M_{f,t,d}^{(\nu)}}. \tag{A.33}$$

Spatial Whitening:

$$\tilde{\mathbf{Y}}_{f,t} = \boldsymbol{\Phi}_{NNf}^{-\frac{1}{2}} \cdot \mathbf{Y}_{f,t} \tag{A.34}$$

where $\boldsymbol{\Phi}_{NNf}^{\frac{1}{2}}$ is the lower triangular matrix from the Cholesky decomposition.

Speaker PSD matrix after Spatial Whitening:

$$\boldsymbol{\Phi}_{\tilde{X}\tilde{X}f} = \frac{\sum_{\check{t}=1}^{T} \sum_{d=1}^{D} M_{f,\check{t},d}^{(X)} \tilde{\mathbf{Y}}_{f,\check{t}} \cdot \tilde{\mathbf{Y}}_{f,\check{t}}^{\mathrm{H}}}{\sum_{\check{t}=1}^{T} \sum_{d=1}^{D} M_{f,\check{t},d}^{(X)}}. \tag{A.35}$$

PCA beamformer after spatial whitening:

$$\mathbf{w}_{f}^{(\mathrm{SW})} = \mathcal{P}\left(\boldsymbol{\Phi}_{\tilde{X}\tilde{X}f}\right). \tag{A.36}$$

GEV beamformer:

$$\mathbf{w}_{f}^{(\mathrm{GEV})} = \frac{\boldsymbol{\Phi}_{NNf}^{-\frac{1}{2}\mathrm{H}} \mathbf{w}_{f}^{(\mathrm{SW})}}{\left\|\boldsymbol{\Phi}_{NNf}^{-\frac{1}{2}\mathrm{H}} \mathbf{w}_{f}^{(\mathrm{SW})}\right\|}. \tag{A.37}$$

Proof. GEV criterion:

$$\mathbf{w}_{f}^{(\mathrm{GEV})} = \operatorname*{argmax}_{\mathbf{w}_{f}} \left\{ \frac{\mathbf{w}_{f}^{\mathrm{H}} \boldsymbol{\Phi}_{XXf} \mathbf{w}_{f}}{\mathbf{w}_{f}^{\mathrm{H}} \boldsymbol{\Phi}_{NNf} \mathbf{w}_{f}} \right\} \tag{A.38}$$

results in the generalized eigenvalue problem

$$\boldsymbol{\Phi}_{XXf} \mathbf{w}_{f}^{(\mathrm{GEV})} = \lambda_{\max} \boldsymbol{\Phi}_{NNf} \mathbf{w}_{f}^{(\mathrm{GEV})}. \tag{A.39}$$

Left multiplying both sides with $\boldsymbol{\Phi}_{NNf}^{-1} = \left(\boldsymbol{\Phi}_{NNf}^{\frac{1}{2}} \boldsymbol{\Phi}_{NNf}^{\frac{1}{2}\mathrm{H}}\right)^{-1} = \boldsymbol{\Phi}_{NNf}^{-\frac{1}{2}\mathrm{H}} \boldsymbol{\Phi}_{NNf}^{-\frac{1}{2}}$ and insert (A.37)

$$\boldsymbol{\Phi}_{NNf}^{-\frac{1}{2}\mathrm{H}} \boldsymbol{\Phi}_{NNf}^{-\frac{1}{2}} \boldsymbol{\Phi}_{XXf} \boldsymbol{\Phi}_{NNf}^{-\frac{1}{2}\mathrm{H}} \mathbf{w}_{f}^{(\mathrm{SW})} = \lambda_{\max} \boldsymbol{\Phi}_{NNf}^{-\frac{1}{2}\mathrm{H}} \mathbf{w}_{f}^{(\mathrm{SW})}, \tag{A.40}$$

$$\underbrace{\boldsymbol{\Phi}_{NNf}^{-\frac{1}{2}} \boldsymbol{\Phi}_{XXf} \boldsymbol{\Phi}_{NNf}^{-\frac{1}{2}\mathrm{H}}}_{\overset{!}{=}\boldsymbol{\Phi}_{\tilde{X}\tilde{X}f}} \mathbf{w}_{f}^{(\mathrm{SW})} = \lambda_{\max} \mathbf{w}_{f}^{(\mathrm{SW})}, \tag{A.41}$$

Using the definition for $\boldsymbol{\Phi}_{\tilde{X}\tilde{X}f}$

$$\boldsymbol{\Phi}_{\tilde{X}\tilde{X}f} = \frac{\sum_{\check{t}=1}^{T} \sum_{d=1}^{D} M_{f,\check{t},d}^{(X)} \boldsymbol{\Phi}_{NNf}^{-\frac{1}{2}} \mathbf{Y}_{f,t} \mathbf{Y}_{f,t}^{\mathrm{H}} \boldsymbol{\Phi}_{NNf}^{-\frac{1}{2}\mathrm{H}}}{\sum_{\check{t}=1}^{T} \sum_{d=1}^{D} M_{f,\check{t},d}^{(X)}} \tag{A.42}$$

$$= \boldsymbol{\Phi}_{NNf}^{-\frac{1}{2}} \boldsymbol{\Phi}_{XX} \boldsymbol{\Phi}_{NNf}^{-\frac{1}{2}\mathrm{H}} \tag{A.43}$$

shows, that both are equal.

$\blacksquare$

Symbols

$(\cdot)^{\mathrm{T}}$	transposed
$(\cdot)^{-\mathrm{T}}$	transposed and invert (swappable)
$(\cdot)^{\mathrm{H}}$	conjugate transposed
$(\cdot)^{-\mathrm{H}}$	conjugate transposed and invert (swappable)
$\lvert \cdot \rvert$	absolute value of a scalar
$\lVert \cdot \rVert$	Euclidean norm of a vector
$\mathrm{Tr}\,(\cdot)$	Trace-Operator
$\cdot \circ \cdot$	Hadamard product or elementwise product
$\Re\{\cdot\}$	Real Part
$\Im\{\cdot\}$	Imaginary Part
j	Complex number
e	Euler's number
δ_{nm}	Kronecker delta
$\mathbf{I}$	Identity matrix
$\mathbf{0}$	vector/matrix filled with zeros
$\mathbf{1}$	vector/matrix filled with ones
$\triangle$	lower triangular matrix filled with ones
$\triangledown$	upper triangular matrix filled with ones
$\forall_{\mathrm{case}}$	for all where case is true
$(\cdot) \in \{\dots\}$	$(\cdot)$ is in the set $\{\dots\}$
$(\cdot) \in \blacktriangle$	a lower triangular matrix
$(\cdot) \in \blacktriangledown$	a upper triangular matrix
d	channel index
n	time index
$t, \check{t}$	time index of STFT
f	frequency index
$\mathbf{X}_{f,t},\ \mathbf{X}_t$	multichannel clean speech in STFT domain

$X_{f,t,1}$	reference speech signal on channel one
$\hat{X}_{f,t,1}$	estimated reference speech signal on channel one
$\mathbf{N}_{f,t}, \mathbf{N}_t$	multichannel noise in STFT domain
$\mathbf{Y}_{f,t}, \mathbf{Y}_t$	multichannel noisy signal in STFT domain
$M^{(X)}_{f,t,d}, M^{(X)}_t$	speech mask (interpretable as speech presence probability)
$M^{(N)}_{f,t,d}, M^{(N)}_t$	noise mask (interpretable as noise presence probability)
$\mathbf{\Phi}_{XXf}$	speech PSD
$\mathbf{\Phi}_{NNf}$	noise PSD
$\mathbf{\Phi}, \mathbf{\Phi}_f$	noise PSD inverse multiplied with speech PSD
$P^{(X)}$	power of (beamformered) speech signal
$P^{(N)}$	power of (beamformered) noise signal
$\mathbf{w}_f, \mathbf{w}$	beamforming vector
$\mathbf{w}^{(\text{MVDR})}_f, \mathbf{w}^{(\text{PCA})}_f, \mathbf{w}^{(\text{GEV})}_f$	MVDR/PCA/GEVbeamforming vector
$\mathbf{W}, \mathbf{W}_f$	Eigenvectors
$\mathbf{\Lambda}, \mathbf{\Lambda}_f$	Eigenvalue matrix
λ	Eigenvalue
$\mathcal{P}(\cdot)$	principal component (eigenvector corresponding to the largest eigenvalue)
z	scalar, input value of a chain
ζ	placeholder for z and z^*
$\mathbf{z}$	vector, input value of a chain
$\mathbf{Z}$	matrix, input value of a chain
x	real part of z
y	imaginary part of z
s	intermediate value of a chain
σ	real part of s
ω	imaginary part of s
J	final real value of a chain, objective function
f, g	function
A, B	matrix, previous intermediate value of a chain
C	matrix, later intermediate value of a chain
$\frac{\partial J}{\partial x}, \frac{\partial J}{\partial y}$	gradient, real scalar, partial derivative of J w.r.t. x or y

$\frac{\partial \sigma}{\partial x}$	real scalar, partial derivative of σ w.r.t. x
$\frac{\partial J}{\partial z^*}$, ∇_{z^*}	gradient, scalar, partial derivative of J w.r.t. z^*
$\frac{\partial J}{\partial z}$, ∇_z	gradient, scalar, partial derivative of J w.r.t. z
$\frac{\partial J}{\partial \mathbf{z}^*}$, $\nabla_{\mathbf{z}^*}$	gradient, column vector, partial derivative of J w.r.t. $\mathbf{z}^*$, shape like $\mathbf{z}$
$\frac{\partial J}{\partial \mathbf{Z}^*}$, $\nabla_{\mathbf{Z}^*}$	gradient, matrix, partial derivative of J w.r.t. $\mathbf{z}^*$, shape like $\mathbf{Z}$
$\frac{\partial \mathbf{C}}{\partial z^*}$	gradient, matrix, partial derivative of $\mathbf{C}$ w.r.t. z^*, shape like $\mathbf{C}$
$\mathbf{E}$	matrix of eigenvalue differences $E_{i,j} = \lambda_j - \lambda_i$
$\mathbf{F}$	matrix of inverse eigenvalue differences $F_{i,j} = (\lambda_j - \lambda_i)$, except on the diagonal where it is zero
$\underset{n \to f}{\mathrm{DFT}} \{\cdot\}$	discrete fourier transformation from time index n to frequency index f
$\underset{f \to n}{\mathrm{IDFT}} \{\cdot\}$	inverse discrete fourier transformation from frequency index f to time index n
$\underset{n \to f}{\mathrm{RDFT}} \{\cdot\}$	discrete fourier transformation for real input from time index n to frequency index f
$\underset{f \to n}{\mathrm{IRDFT}} \{\cdot\}$	inverse discrete fourier transformation for real input from frequency index f to time index n

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Acronyms

AD algorithmic differentiation

ASR automatic speech recognition

BAN blind analytic normalization

BCE binary cross entropy

DFT discrete fourier transformation

GEV generalized eigenvector

IBM ideal binary mask

IDFT inverse discrete fourier transformation

IRDFT inverse discrete fourier transformation for real input

ML maximum likelihood

MVDR minimum variance distortionless response

NN neural network

PESQ perceptual evaluation of speech quality [Rix+01]

PSD power spectral density

RDFT discrete fourier transformation for real input

SNR signal-to-noise ratio

STFT short-time Fourier transform

WER word error rate

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